# Configurations and colouring problems in block designs

## Thesis




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Anthony Duncan Forbes
B.A. (Hons.)

# Configurations and colouring problems in block designs

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Department of Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

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## Abstract

A Steiner triple system of order $v$ (STS($v$)) is called $\chi$-*chromatic* if $\chi$ is the smallest number of colours needed to avoid monochromatic blocks. Amongst our results on colour class structures we show that every STS(19) is 3- or 4-chromatic, that every 3-chromatic STS(19) has an *equitable* 3-colouring (meaning that the colours are as uniformly distributed as possible), and that for all admissible $v \geq 25$ there exists a 3-chromatic STS($v$) which does not admit an equitable 3-colouring. We obtain a formula for the number of independent sets in an STS($v$) and use it to show that an STS(21) must contain eight independent points. This leads to a simple proof that every STS(21) is 3- or 4-chromatic.

Substantially extending existing tabulations, we provide an enumeration of STS trades of up to 12 blocks, and as an application we show that any pair of STS(15)s must be $3^{-1}$-isomorphic.

We prove a general theorem that enables us to obtain formulae for the frequencies of occurrence of configurations in triple systems. Some of these are used in our proof that for $v \geq 25$ no STS($v$) has a 3-existentially closed block intersection graph. Of specific interest in connection with a conjecture of Erdős are 6-sparse and perfect Steiner triple systems, characterized by the avoidance of specific configurations. We describe two direct constructions that produce 6-sparse STS($v$)s and we give a recursive construction that preserves 6-sparseness. Also we settle an old question concerning the occurrence of perfect block transitive Steiner triple systems.

Finally, we consider Steiner $S(2,4,v)$ designs that are built from collections of Steiner triple systems. We solve a longstanding problem by constructing such systems with $v = 61$ (Zoe's design) and $v = 100$ (the design of the century).

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## List of publications

A. D. Forbes, Uniquely 3-colourable Steiner triple systems, *J. Combin. Theory, Series A* **101** (2003), 49–68.

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# Contents

# Chapter 1

# Basic concepts

## 1.1 Block designs

In this thesis we are concerned with properties of a class of combinatorial block designs known as Steiner systems. Whilst it is not our purpose to provide a comprehensive account of block designs in general, we collect together in this introductory chapter definitions and notational conventions that will recur throughout the remainder of the work.

A *Steiner system*, $S(t,k,v)$, is a pair $(V,\mathcal{B})$, where $V$ is a finite set of cardinality $v$ of *elements*, or *points*, and $\mathcal{B}$ is a collection of $k$-element subsets of $V$, called *blocks*, which has the property that every $t$-element subset of $V$ occurs in precisely one block. We can also consider a more general structure, denoted by $S_\lambda(t,k,v)$, a pair $(V,\mathcal{B})$ where $|V| = v$ and $\mathcal{B}$ is a collection of $k$-element subsets of $V$ such that every $t$-element subset of $V$ occurs in precisely $\lambda$ blocks. However, we would need some flexibility in the meaning of the word 'collection' to allow for the possibility of repeated blocks when $\lambda > 1$. Indeed, to handle these and even more general designs it is customary to introduce the concept of an *incidence structure*—that is, a triple $(V,\mathcal{B},\iota)$, where $V$ (the points) and $\mathcal{B}$ (the blocks) are two disjoint sets and $\iota \subseteq V \times \mathcal{B}$ is an *incidence relation* that associates points with blocks.

An important special case, and indeed one which will occupy by far the major part of this thesis, is where $t = 2$ and $k = 3$. An $S(2,3,v)$ is called a *Steiner triple system of order* $v$, or STS($v$) for short. Steiner triple systems exist if and only if $v \equiv 1$ or $3 \pmod 6$ [45]. Such values of $v$ are called *admissible*.

The notation just described will be used throughout this work. However, there are alternative definitions in the literature. A $(v,k,\lambda)$ *balanced incomplete block design*, or $(v,k,\lambda)$ BIBD, is an $S_\lambda(2,k,v)$, and the more general $S_\lambda(t,k,v)$ is called a $t$-design and denoted by $t$-$(v,k,\lambda)$.

Two STS$(v)$s, $S=(V,\mathcal{B})$ and $S'=(V',\mathcal{B}')$ are *isomorphic*, $S\cong S'$, if there exists a bijection $\phi: V\to V'$ such that for each block $T\in\mathcal{B}$, $\phi(T)$ is a block in $\mathcal{B}'$. (We adopt the usual convention of writing $f(X)$ for $\{f(x):x\in X\}$ if $f$ is a function and $X$ is a subset of its domain.) The bijection $\phi$ is called an *isomorphism* from $S$ to $S'$. Up to isomorphism, the STS(1), STS(3), STS(7) and STS(9) are unique. There are two STS(13)s, 80 STS(15)s and, as shown by Kaski and Östergård [43], 11,084,874,829 STS(19)s. The corresponding numbers for larger $v$ are unknown.

Let $S=(V,\mathcal{B})$ be an STS$(v)$. A permutation $\phi: V\mapsto V$ of the points of $S$ which preserves the block structure, in the sense that $B\in\mathcal{B}\Leftrightarrow\phi(B)\in\mathcal{B}$, is called an *automorphism* of $S$. The set of all automorphisms of $S$ together with the operation of composition forms a group, called the *full automorphism group* of $S$ and is denoted by $\mathrm{Aut}(S)$.

A *sub-STS*$(u)$, $(U,\mathcal{A})$, of an STS$(v)$, $(V,\mathcal{B})$, is a Steiner triple system of order $u$ where $U$ is a $u$-element subset of $V$ and $\mathcal{A}=\{T\in\mathcal{B}:T\subseteq U\}$. It is well known that $u\le(v-1)/2$ and that when equality holds $V\setminus U$ is an independent set, as defined in section 1.4 [19].

## 1.2 Colourings

Broadly speaking, a 'colouring' of a block design is the assignment of a colour to each point of the design together with a rule which restricts the colours that can appear in a block. For Steiner triple systems, a rule which has generated considerable interest is that blocks must not be monochromatic. Each block must contain either two points of one colour and one point of a different colour, or three points of distinct colours. For these systems, this is the only rule we shall consider in this thesis. However, there is an interesting alternative. Indeed, the rule that precisely two colours must be present in every block is the subject of a paper by Colbourn, Dinitz & Rosa [11].

In Chapter 11, where the primary objects of study are $S(2,4,v)$s, we consider a different colouring rule: blocks must be bichromatic, with three points of one colour and one point of a different colour. Again, this is one of a number of interesting colouring rules [54]. But our particular motivation is, as we shall see, a close connection between this (3, 1) colouring of $S(2, 4, v)$s and Steiner triple systems.

We now give a formal definition that will be used in Chapters 2 and 3. Let $S = (V, \mathcal{B})$ be a Steiner triple system of order $v$. An *$m$-colouring* of $S$ is a surjection $\phi : V \to C$, where $C$ is a set of cardinality $m$, whose elements are called *colours*, such that $|\phi(T)| > 1$ for all $T \in \mathcal{B}$. Thus no block is monochromatic. If $S$ has an $m$-colouring, we say that $S$ is *$m$-colourable.* We define $S$ to be *$\chi$-chromatic* if $S$ is $\chi$-colourable but not $(\chi - 1)$-colourable; if this is the case, $\chi$ is called the *chromatic number* of $S$. It is obvious that $S$ is $(m + 1)$-colourable if $S$ is $m$-colourable and $m < v$.

## 1.3 Configurations

In the context of Steiner triple systems, an $n$-block *configuration* is a collection of $n$ triples, called blocks, which has the property that every pair of distinct elements occurs in at most one block.

In discussing configurations and Steiner triple systems, it is extremely useful to have a compact notation for the third point of a block in terms of the other two. So throughout this work we denote by $x * y$ the third point in a block containing $x$ and $y$. When the defining configuration is the set of blocks of a Steiner triple system with point set $V$, the operation $* : V \times V \mapsto V$, extended by setting $x * x = x$, is a *quasi-group*. This is because the operation $*$ is closed and symmetric, and if $x, y \in V$, then in each of the equations $x * y = z$, $x * z = y$ and $y * z = x$ the variable $z \in V$ is uniquely determined by $x$ and $y$.

If $\mathcal{C}$ is a configuration, we denote the number of blocks by $b(\mathcal{C})$, the number of points by $p(\mathcal{C})$ and the set of points by $P(\mathcal{C})$. The *degree* of a point is the number of blocks which contain it.

Two configurations $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, if there exists a one-to-one mapping $\phi : P(\mathcal{C}) \to P(\mathcal{D})$ such that for each block $T \in \mathcal{C}$, $\phi(T)$ is a block in $\mathcal{D}$. An *automorphism* of a configuration $\mathcal{C}$ is a permutation $\phi : P(\mathcal{C}) \to P(\mathcal{C})$

such that $T \in \mathcal{C} \Leftrightarrow \phi(T) \in \mathcal{C}$. The full automorphism group of $\mathcal{C}$ is denoted by $\mathrm{Aut}(\mathcal{C})$.

## 1.4 Independent sets

A subset $I$ of $V$ in an STS($v$), $S = (V, \mathcal{B})$, is an *independent set* if no three points of $I$ occur in a block. An independent set in $S$ is *maximal* if for all $x \in V \setminus I$, $I \cup \{x\}$ is not an independent set in $S$, and *maximum* if it has the largest possible cardinality of any independent set in $S$.

As is well known, if $I$ is a maximum independent set in $S$, then $|I| \leq (v + \epsilon)/2$, where $\epsilon = 1$ if $v \equiv 3$ or $7 \pmod{12}$ and $\epsilon = -1$ if $v \equiv 1$ or $9 \pmod{12}$. In the former case, if $I$ has cardinality $(v+1)/2$, then the complement $V^* = V \setminus I$ together with $\mathcal{B}^*$, the blocks of $\mathcal{B}$ whose elements belong to $V^*$, is a Steiner triple system of order $(v-1)/2$, $(V^*, \mathcal{B}^*)$ [65].

# Chapter 2

# Colourings of Steiner triple systems

## 2.1 Introduction

Much of the original work on colourings of Steiner triple systems is due to Rosa [62], [63]. The unique STS(3) is 2-chromatic, and is the only STS($v$) which is. Steiner triple systems which are 3-chromatic exist for all $v \equiv 1$ or 3 (mod 6), $v \geq 7$, (using the Bose and Skolem constructions; see [63] or section 2.6 of this thesis). The STS(7), the STS(9), the two STS(13)s and the 80 STS(15)s are 3-chromatic [53]. But more generally it is known that if $m \geq 3$, then for all sufficiently large admissible $v$, there exists an $m$-chromatic STS($v$) [16, Theorem 18.6]. Currently the only value of $v$ for which the existence of a 4-chromatic STS($v$) is in doubt is $v = 19$ [6, 39].

In this chapter we are mainly interested in 3-chromatic STS($v$)s and in particular the distribution of the colour class sizes. Thus we define an STS($v$) to be *equitably $m$-chromatic* if it is $m$-chromatic and the cardinalities of the colour classes differ by at most one. This concept was introduced by Rosa in [64]. Although it is known that for every $m \geq 6$, there exists an $m$-chromatic STS($v$) which does not admit an equitable $m$-colouring [40], the cases $m = 3$, 4 and 5 are in doubt. In fact, one of the work points at the end of Chapter 18 of [16] specifically asks whether every 3-chromatic STS($v$) admits an equitable 3-colouring. Also at the beginning of section 18.2 of [16] it is suggested that they do. However, in Chapter 3 we show that 3-chromatic STS($v$) which do not admit equitable 3-colourings exist for all admissible $v \geq 25$ (Theorem 3.1.2).

## 2.2 The cases $v \leq 15$

Let $S = (V, B)$ be a 3-chromatic STS($v$). Both in this section and throughout the remainder of the chapter we will denote the colour classes by $R$ (red), $Y$ (yellow) and $B$ (blue), where $|R| = c_1 \geq |Y| = c_2 \geq |B| = c_3$. The following lemma is due to Haddad and Rődl [40].

**Lemma 2.2.1** $v = c_1 + c_2 + c_3 \geq ((c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2)/2.$

**Proof.** See the original paper and also [16, page 331]. □

The next lemma gives an upper bound for the size of a colour class.

**Lemma 2.2.2** *If $v > 7$ then $c_1 \leq (v-1)/2$.*

**Proof.** Let $r = (v-1)/2$. A given red point can be paired with the remaining $c_1 - 1$ red points in at most $r$ blocks. Hence $c_1 \leq r + 1$. Now suppose that $c_1 = r + 1 = (v+1)/2$. Pairs of distinct points $x, y \in R$ appear in different blocks and hence there are $(r+1)r/2 = (v+1)(v-1)/8$ such blocks. There are $(v-1)/2$ points which are not in the colour class $R$. Hence the number of pairs $\{x, l\}, x \in R, l \notin R$ is $(v+1)(v-1)/4$ and these must all appear in the blocks described above. The remaining blocks of the STS($v$) must therefore comprise a sub-STS($(v-1)/2$) and moreover this subsystem must be 2-chromatic. Thus $(v-1)/2 = 3$ and hence $v = 7$. □

### The case $v = 7$

Let the unique STS(7) be given by $V = \{0, 1, \ldots, 6\}$ and $\mathcal{B} = \{\{0,1,3\}, \{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,0\}, \{5,6,1\}, \{6,0,2\}\}$. The possible colouring patterns are $(c_1, c_2, c_3) = (4,2,1), (3,3,1)$ or $(3,2,2)$ and all are attainable as follows.

1. $(c_1, c_2, c_3) = (4,2,1)$; $R = \{0,1,2,5\}, Y = \{3,4\}, B = \{6\}$.
2. $(c_1, c_2, c_3) = (3,3,1)$; $R = \{0,1,2\}, Y = \{3,4,5\}, B = \{6\}$.
3. $(c_1, c_2, c_3) = (3,2,2)$; $R = \{0,1,2\}, Y = \{3,4\}, B = \{5,6\}$.

**The case $v = 9$**

Let the unique STS(9) be given by $V = \{0, 1, \ldots, 8\}$ and $\mathcal{B} = \{\{0,1,2\}, \{3,4,5\}, \{6,7,8\}, \{0,3,6\}, \{1,4,7\}, \{2,5,8\}, \{0,4,8\}, \{1,5,6\}, \{2,3,7\}, \{0,5,7\}, \{1,3,8\}, \{2,4,6\}\}$. The possible colouring patterns are $(c_1, c_2, c_3) = (4,4,1), (4,3,2)$ or $(3,3,3)$ and all are attainable as follows.

1. $(c_1, c_2, c_3) = (4,4,1)$; $R = \{0,1,3,4\}, Y = \{2,5,6,7\}, B = \{8\}$.

2. $(c_1, c_2, c_3) = (4,3,2)$; $R = \{0,1,3,4\}, Y = \{2,5,6\}, B = \{7,8\}$.

3. $(c_1, c_2, c_3) = (3,3,3)$; $R = \{0,1,3\}, Y = \{2,4,7\}, B = \{5,6,8\}$.

**The case $v = 13$**

The cyclic STS(13) is given by $V = \{0, 1, \ldots, 12\}$ and $\mathcal{B}$ is the set of 26 blocks generated from $\{0,1,4\}$ and $\{0,2,7\}$ by $i \mapsto i+1 \pmod{13}$. The second STS(13) is formed by replacing the blocks $\{0,1,4\}, \{0,2,7\}, \{2,4,9\}, \{7,9,1\}$ in the above system by the blocks $\{2,7,9\}, \{1,4,9\}, \{0,1,7\}, \{0,2,4\}$. The possible colouring patterns are $(c_1, c_2, c_3) = (6,5,2), (6,4,3), (5,5,3)$ or $(5,4,4)$ and all are attainable for both systems as follows.

1. $(6,5,2)$: $R = \{1,2,4,6,7,8\}, Y = \{5,9,10,11,12\}, B = \{0,3\}$.

2. $(6,4,3)$: $R = \{1,2,4,6,7,8\}, Y = \{9,10,11,12\}, B = \{0,3,5\}$.

3. $(5,5,3)$: $R = \{0,1,2,3,9\}, Y = \{4,5,7,10,11\}, B = \{6,8,12\}$.

4. $(5,4,4)$: $R = \{0,1,2,3,9\}, Y = \{4,5,7,10\}, B = \{6,8,11,12\}$.

**The case $v = 15$**

The possible colouring patterns are $(c_1, c_2, c_3) = (7,5,3)$, $(7,4,4)$, $(6,6,3)$, $(6,5,4)$ or $(5,5,5)$. The case of the STS(15) formed by the point-line design of the projective geometry PG(3,2) (#1 in the listing of [53]) was considered by J. Pelikán [57]. Every 3-colouring of this system is equitable, i.e. it is 3-chromatic but only with colour classes of cardinality $(c_1, c_2, c_3) = (5, 5, 5)$. We have determined that 76 of the 80 STS(15)s can be 3-coloured with any of the 5 colouring patterns listed above. But system #7 is 3-chromatic only with colour classes $(c_1, c_2, c_3) = (6, 5, 4)$ or $(5, 5, 5)$ and systems #79 and #80 are 3-chromatic only with colour classes $(c_1, c_2, c_3) = (6, 6, 3), (6, 5, 4)$ or $(5, 5, 5)$.

## 2.3 Non-equitably 3-colourable STS($v$)s

In this section we present some 3-chromatic STS($v$)s which do not admit equitable 3-colourings for $v$ = 25, 27, 31, 33, 37 and 39. All systems were constructed by 'hill-climbing' [68]. As far as the author is aware, STS(27) #1 has the historical distinction of being the first such Steiner triple system to be found. In fact, this (accidental) discovery and the uniqueness of its (non-equitable) 3-colouring ultimately led to the more systematic account in Chapter 3. The other systems are further sporadic discoveries of the author during the early days of his researches into 3-colourings of Steiner triple systems.

We use the same compact notation as described in [14] and [37]. The base set is $V = \{0, 1, 2, \ldots, v-1\}$ and the blocks are represented by a string of symbols $s_1, s_2, \ldots, s_b$, where $b = v(v-1)/6$. Using the usual lexicographical order, the symbol $s_i$ is the largest element $z_i$ in the $i$-th triple $\{x_i, y_i, z_i\}$ where $x_i < y_i < z_i$. The sequence of blocks is reconstructed in a straightforward manner. The first block is $\{0, 1, s_1\}$. Thereafter, assuming that $n-1$ blocks have been reconstructed, $n = 2, 3, \ldots, b$, the $n$th block is $\{a_n, b_n, s_n\}$, where the $\{a_n, b_n\}$ is the smallest pair in the lexicographical ordering that is not present in the first $n-1$ blocks. It is convenient to use a single character for each symbol; so integers greater than 9 are represented by letters:

```
a=10, b=11, c=12, d=13, e=14, f=15, g=16, h=17, i=18, j=19,
k=20, l=21, m=22, n=23, o=24, p=25, q=26, r=27, s=28, t=29,
u=30, v=31, w=32, x=33, y=34, z=35, A=36, B=37, C=38, D=39,
E=40, F=41, G=42, H=43, I=44, J=45, K=46, L=47, M=48, N=49,
O=50, P=51, Q=52, R=53, S=54, T=55, U=56, V=57, W=58, X=59.
```

For example, the first two blocks of the first system that follows are $\{0, 1, 21\}$ and $\{0, 2, 4\}$. There should be no confusion between symbols of similar appearance because the numbers zero and one never occur.

**STS(25)**

`149mijkbfo hn9abgjhin kfobakilhn gomnocdeij kmldecomjk`

`iecdbkjnnm fglooglfmh fgnnkjmljh iooijmnelm lmlohnonko`

The 3-colouring is unique, modulo swapping colour classes $R$ and $Y$, with pattern $(c_1, c_2, c_3) = (9, 9, 7)$ and is given by $R = \{0,\ 1,\ 2,\ \text{a},\ \text{h},\ \text{i},\ \text{j},\ \text{n},\ \text{o}\}$, $Y = \{3,\ 4,\ 5,\ \text{b},\ \text{c},\ \text{e},\ \text{f},\ \text{g},\ \text{m}\}$, $B = \{6,\ 7,\ 8,\ 9,\ \text{d},\ \text{k},\ \text{l}\}$. The number of blocks of each colour type is $RRY = 18$, $RRB = 18$, $RYY = 19$, $RBB = 10$, $YYB = 17$, $YBB = 11$, $RYB = 7$. The system has trivial full automorphism group.

**STS(27) #1**

`n9pmd8blej oiqco6ga9f qjplm6ko8d ignjpqfbkd ighqnmpngh`

`ieqjoh9iqj mlnobcnlmp qkfgilqpjm ofqpnekpon mqkpnonohl`

`qlmpkimopq kjlonpq`

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (10, 9, 8)$ and is given by $R = \{4,\ 8,\ 9,\ \text{b},\ \text{f},\ \text{i},\ \text{j},\ \text{k},\ \text{n},\ \text{q}\}$, $Y = \{3,\ 5,\ 7,\ \text{a},\ \text{c},\ \text{d},\ \text{e},\ \text{o},\ \text{p}\}$, $B = \{0,\ 1,\ 2,\ 6,\ \text{g},\ \text{h},\ \text{l},\ \text{m}\}$. The number of blocks of each colour type is $RRY = 22$, $RRB = 23$, $RYY = 19$, $RBB = 13$, $YYB = 17$, $YBB = 15$, $RYB = 8$. The system has trivial full automorphism group.

**STS(27) #2**

`k56bedjclo mpq4mal8ij ogqnpiefpm nkhljqdhoc pfenql7neo`

`qkglmpqlbg kopjnbmajp okikjdiqnp qbihonnfhl hglpoiohmq`

`ppqmnqmmkl kqnooqp`

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (10, 9, 8)$ and is given by $R = \{0,\ 1,\ 2,\ 8,\ \text{b},\ \text{e},\ \text{g},\ \text{i},\ \text{l},\ \text{n}\}$, $Y = \{5,\ 9,\ \text{a},\ \text{c},\ \text{d},\ \text{j},\ \text{o},\ \text{p},\ \text{q}\}$, $B = \{3,\ 4,\ 6,\ 7,\ \text{f},\ \text{h},\ \text{k},\ \text{m}\}$. The number of blocks of each colour type is $RRY = 24$, $RRB = 21$, $RYY = 17$, $RBB = 15$, $YYB = 19$, $YBB = 13$, $RYB = 8$. The fact that both systems #1 and #2 are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic. The system has trivial full automorphism group.

**STS(31)**

`tlkcmrbniu sojpqoahfd mpljgksurj 7kfmcqeuts prqtprdsmu`
`gniobeiamp ntlsucsled iunqrijuls kmotgohjtu kqpqkfhtus`
`drmjntcrgj stphlrqtur pqktosjlon qphnsqoutq ruoptrusup rtuns`

The 3-colouring is unique, modulo swapping colour classes $R$ and $Y$, with pattern $(c_1, c_2, c_3) = (11, 11, 9)$ and is given by $R = \{0, 3, 7, 8, 9, \text{a}, \text{e}, \text{f}, \text{l}, \text{m}, \text{p}\}$, $Y = \{2, 4, 6, \text{b}, \text{c}, \text{h}, \text{i}, \text{j}, \text{o}, \text{q}, \text{u}\}$, $B = \{1, 5, \text{d}, \text{g}, \text{k}, \text{n}, \text{r}, \text{s}, \text{t}\}$. The number of blocks of each colour type is $RRY = 26$, $RRB = 29$, $RYY = 30$, $RBB = 16$, $YYB = 25$, $YBB = 20$, $RYB = 9$. The system has trivial full automorphism group.

**STS(33) #1**

`epd8rniqvk lsjowuogva sckltmuwqr phqlawemsu ijtvrk7ole`
`pjwrntsvue ohmfplnswt pmwnvkjtqs ovtjmqrwui lgksdhlprt`
`uudqfpjwsv rpdnotvcth wluoqvlruw gtmsropgnv qtwokquswp`
`mvvrspkuqw notutwsrvw vuvrwu`

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (12, 11, 10)$ and is given by $R = \{4, 6, \text{a}, \text{b}, \text{d}, \text{h}, \text{k}, \text{p}, \text{q}, \text{s}, \text{u}, \text{v}\}$, $Y = \{0, 2, 3, 7, 8, 9, \text{c}, \text{j}, \text{o}, \text{r}, \text{t}\}$, $B = \{1, 5, \text{e}, \text{f}, \text{g}, \text{i}, \text{l}, \text{m}, \text{n}, \text{w}\}$. The number of blocks of each colour type is $RRY = 35$, $RRB = 31$, $RYY = 26$, $RBB = 24$, $YYB = 29$, $YBB = 21$, $RYB = 10$. The system has trivial full automorphism group.

**STS(33) #2**

`8qhaoclifj srntuwvj7u tcbkwrpims qms6rjgdln kwuotgfnts`
`pciqlvuwkh wjetumpnrv wghtdvnmls r9uwjmqios pvdosjfmnv`
`pubvlihrnt qrwtnovqup lsqkmuqtko wrvroukswp vpiltshruj`
`qvtqvworwv utrstqvwuw wsvpwu`

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (12, 11, 10)$ and is given by $R = \{4, 7, a, b, d, g, j, k, l, r, v, w\}$, $Y = \{1, 2, 6, 9, e, f, h, m, o, p, s\}$, $B = \{0, 3, 5, 8, c, i, n, q, t, u\}$. The number of blocks of each colour type is $RRY = 33$, $RRB = 33$, $RYY = 28$, $RBB = 22$, $YYB = 27$, $YBB = 23$, $RYB = 10$. The fact that both systems #1 and #2 are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic. The system has trivial full automorphism group.

**STS(37)**

```
isjekfcgmh nAozyxvwje wflnmhqcpk uzyxAkfcgm onbhqlrzyx
Algmhncudr otszyAmhdc injsovutzA nlidjeryxw vuAidjekAp
zyxwvjAkfg rqzyxwkflA tsrzyxlgvo utszymvrAx wtzAvtzyxw
uupvqwrxsy ztrqwxsytz owxsytzoux sytzAupytz upvAzoupvq
upqwvqwrAw rxAxsvyAAv wA
```

The 3-colouring is unique, modulo swapping colour classes $R$ and $Y$, with pattern $(c_1, c_2, c_3) = (13, 13, 11)$ and is given by $R = \{0, 2, c, e, f, g, h, i, j, k, m, n, A\}$, $Y = \{1, 3, 4, 5, 6, 7, 8, 9, a, b, o, t, v\}$, $B = \{d, l, p, q, r, s, u, w, x, y, z\}$. The number of blocks of each colour type is $RRY = 35$, $RRB = 43$, $RYY = 44$, $RBB = 23$, $YYB = 34$, $YBB = 32$, $RYB = 11$. The system has trivial full automorphism group.

**STS(39)**

```
kelfmgnhoi pjxwCBAzyl fmgnhoipjd rsCBAzymgn hoipjdksut
CBAznhqipj wkoetvuCBA oipjdkeluy xwvCBpjdke lfvAzyxwCd
kelfmwCBAz yxelfwgxyr CBAzfcnmyt sCBAzrhnuz vtCBAgomAx
vuCBitBzyx vCvACBwzyx wysztAuBvr CysztAuBxC wqztAuBvCr
qxqnBvCxwA BvCwxryCmq ysuxqxrysz rsztysztAq AuwuBwv
```

This system has two 3-colourings, both with pattern $(c_1, c_2, c_3) = (14, 13, 12)$.

The first of these is given by $R = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{a}, \mathrm{b}, \mathrm{u}, \mathrm{x}\}$, $Y = \{\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{j}, \mathrm{k}, \mathrm{l}, \mathrm{m}, \mathrm{n}, \mathrm{o}, \mathrm{p}\}$, $B = \{\mathrm{i}, \mathrm{q}, \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{v}, \mathrm{w}, \mathrm{y}, \mathrm{z}, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$. The number of blocks of each colour type is $RRY = 62$, $RRB = 29$, $RYY = 23$, $RBB = 49$, $YBB = 55$, $YBB = 17$, $RYB = 12$. The second colouring is obtained by moving the element b from the colour class $R$ to the colour class $Y$ (and re-naming $R$ as $Y$ and vice-versa). It is given by $R = \{\mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{j}, \mathrm{k}, \mathrm{l}, \mathrm{m}, \mathrm{n}, \mathrm{o}, \mathrm{p}\}$, $Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{a}, \mathrm{u}, \mathrm{x}\}$, $B = \{\mathrm{i}, \mathrm{q}, \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{v}, \mathrm{w}, \mathrm{y}, \mathrm{z}, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$. The number of blocks of each colour type is $RRY = 33$, $RRB = 58$, $RYY = 52$, $RBB = 20$, $YYB = 26$, $YBB = 46$, $RYB = 12$. The system has trivial full automorphism group.

## 2.4 The cases $v = 19$ and 21

The results of the preceding section leave open the existence of STS(19)s and STS(21)s which do not admit equitable 3-colourings. Here we show that every 3-colourable STS(19) has an equitable 3-colouring (with colour class sizes 7, 6 and 6). We begin with three lemmas.

**Lemma 2.4.1** *Suppose there exists a 3-chromatic STS(v) with colour classes R, Y, B as above of cardinalities* $c_1, c_2, c_3$ *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_1 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1 - 1, c_2, c_3 + 1$.

**Proof.** There are exactly $\binom{c_3}{2}$ blocks of the form $\{x, l, m\}, l \in B, m \in B$. Thus if $c_1 > \binom{c_3}{2}$ then for some $z \in R$, there is no block of the form $\{z, l, m\}, l \in B, m \in B$. The point $z$ may then be recoloured blue. □

**Lemma 2.4.2** *Suppose there exists a 3-chromatic STS(v) with colour classes R, Y, B as above of cardinalities* $c_1, c_2, c_3$ *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_2 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1, c_2 - 1, c_3 + 1$.

**Proof.** As Lemma 2.4.1. □

**Lemma 2.4.3** *Suppose there exists a 3-chromatic STS(v) with colour classes R, Y, B as above of cardinalities* $c_1, c_2, c_3$ *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_1 + c_2 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1 - 1, c_2, c_3 + 1$ *or* $c_1, c_2 - 1, c_3 + 1$.

**Proof.** By the same argument as used in Lemma 2.4.1, there are exactly $\binom{c_3}{2}$ blocks of the form $\{x, l, m\}, l \in B, m \in B$. Thus if $c_1 + c_2 > \binom{c_3}{2}$ then for some $z \in R \cup Y$, there is no block of the form $\{z, l, m\}, l \in B, m \in B$. The point $z$ may then be recoloured blue. □

In the next two theorems we will use the notation $(c_1, c_2, c_3) \to (c_1', c_2', c_3')$ to denote that a 3-colouring of an STS($v$) with colour classes of cardinalities $c_1, c_2, c_3$ can be transformed into a 3-colouring with colour classes of cardinalities $c_1', c_2', c_3'$.

**Theorem 2.4.1** *Every 3-chromatic STS(19) is equitably 3-colourable.*

**Proof.** The possible colouring patterns, as determined by Lemma 2.2.1, are $(c_1, c_2, c_3) = (9, 6, 4)$ or $(9, 5, 5)$ or $(8, 7, 4)$ or $(8, 6, 5)$ or $(7, 7, 5)$ or $(7, 6, 6)$. Let $x_{ijk}$ denote the number of blocks containing points of colour classes $c_i$, $c_j$ and $c_k$ (with the appropriate multiplicities of the subscripts), $1 \leq i \leq j \leq k \leq 3$. Write $x$ for $x_{223}$. By a straightforward computation we can construct the following table.

| $c_1\ c_2\ c_3$ | $x_{122}$ | $x_{133}$ | $x_{112}$ | $x_{113}$ | $x_{223}$ | $x_{233}$ | $x_{123}$ |
|---|---|---|---|---|---|---|---|
| 7 6 6 | $15 - x$ | $x$ | $3 + x$ | $18 - x$ | $x$ | $15 - x$ | 6 |
| 7 7 5 | $21 - x$ | $-5 + x$ | $1 + x$ | $20 - x$ | $x$ | $15 - x$ | 5 |
| 8 6 5 | $15 - x$ | $-3 + x$ | $7 + x$ | $21 - x$ | $x$ | $13 - x$ | 4 |
| 8 7 4 | $21 - x$ | $-7 + x$ | $6 + x$ | $22 - x$ | $x$ | $13 - x$ | 2 |
| 9 5 5 | $10 - x$ | $-2 + x$ | $12 + x$ | $24 - x$ | $x$ | $12 - x$ | 1 |
| 9 6 4 | $15 - x$ | $-6 + x$ | $12 + x$ | $24 - x$ | $x$ | $12 - x$ | 0 |

We prove the theorem by showing that a 3-colouring with any of these colouring patterns can be transformed into a 3-colouring with $(c_1, c_2, c_3) = (7, 6, 6)$.

Either $(9, 5, 5) \to (9, 6, 4)$ or $(9, 5, 5) \to (8, 6, 5)$ by Lemma 2.4.3.

$(9, 6, 4) \to (8, 6, 5)$ by Lemma 2.4.1.

$(8, 7, 4) \to (7, 7, 5)$ by Lemma 2.4.1.

$(7, 7, 5) \to (7, 6, 6)$ by Lemma 2.4.3.

To complete the proof we show that either $(8, 6, 5) \to (7, 7, 5)$ or $(8, 6, 5) \to (7, 6, 6)$. If $x_{122} \leq 7$, we can find a red point whose colour can be changed to yellow without creating a monochromatic yellow block. Similarly, if $x_{133} \leq 7$ then we can find a red point to change to blue without creating a monochromatic blue block. But from the table we see that $x_{122} \geq 8$ and $x_{133} \geq 8$ imply $11 \leq 3 + x_{133} = x = 15 - x_{122} \leq 7$, a contradiction. □

**Theorem 2.4.2** *Every 3-chromatic STS(21) has a 3-colouring in which the cardinalities of the colour classes are* $(c_1, c_2, c_3) = (8, 7, 6)$ *or* $(7, 7, 7)$.

**Proof.** The possible colouring patterns, as determined by Lemma 2.2.1, are $(c_1, c_2, c_3) = (10, 6, 5)$ or $(9, 8, 4)$ or $(9, 7, 5)$ or $(9, 6, 6)$ or $(8, 8, 5)$ or $(8, 7, 6)$ or $(7, 7, 7)$. Let $x_{ijk}$ denote the number of blocks containing points of colour classes $c_i$, $c_j$ and $c_k$ (with the appropriate multiplicities), $1 \leq i \leq j \leq k \leq 3$. Write $x$ for $x_{223}$. By a straightforward computation we can construct the following table.

| $c_1$ $c_2$ $c_3$ | $x_{122}$ | $x_{133}$ | $x_{112}$ | $x_{113}$ | $x_{223}$ | $x_{233}$ | $x_{123}$ |
|---|---|---|---|---|---|---|---|
| 7 7 7 | $21 - x$ | $x$ | $x$ | $21 - x$ | $x$ | $21 - x$ | 7 |
| 8 7 6 | $21 - x$ | $-3 + x$ | $4 + x$ | $24 - x$ | $x$ | $18 - x$ | 6 |
| 8 8 5 | $28 - x$ | $-8 + x$ | $2 + x$ | $26 - x$ | $x$ | $18 - x$ | 4 |
| 9 6 6 | $15 - x$ | $-1 + x$ | $10 + x$ | $26 - x$ | $x$ | $16 - x$ | 4 |
| 9 7 5 | $21 - x$ | $-6 + x$ | $9 + x$ | $27 - x$ | $x$ | $16 - x$ | 3 |
| 9 8 4 | $28 - x$ | $-10 + x$ | $8 + x$ | $28 - x$ | $x$ | $16 - x$ | 0 |
| 10 6 5 | $15 - x$ | $-5 + x$ | $15 + x$ | $30 - x$ | $x$ | $15 - x$ | 0 |

We first show that either $(10, 6, 5) \to (9, 7, 5)$ or $(10, 6, 5) \to (9, 6, 6)$. If $x_{122} \leq 9$ or $x_{133} \leq 9$ we can find a red point to change to yellow or blue respectively. On the other hand, $x_{122} \geq 10$ and $x_{133} \geq 10$ imply $15 \leq 5 + x_{133} = x = 15 - x_{122} \leq 5$, a contradiction.

Next, either $(9, 7, 5) \to (9, 6, 6)$ or $(9, 7, 5) \to (8, 7, 6)$ by Lemma 2.4.3.

Also $(9, 8, 4) \to (8, 8, 5)$ by Lemma 2.4.1 and $(8, 8, 5) \to (8, 7, 6)$ by Lemma 2.4.3.

To complete the proof we show that $(9, 6, 6) \to (8, 7, 6)$. If $x_{122} \leq 8$ or $x_{133} \leq 8$ we can find a red point to change to yellow or blue respectively. On the other hand, $x_{122} \geq 9$ and $x_{133} \geq 9$ imply $10 \leq 1 + x_{133} = x = 15 - x_{122} \leq 6$, a contradiction. □

At the beginning of this chapter we remarked that the only value of $v$ for which the existence of a 4-chromatic STS($v$) is in doubt is $v = 19$. However, 4-chromatic STS(21)s exist. The first 4-chromatic STS(21) was constructed by Haddad [39] and plays a significant role in determining the spectrum of 4-chromatic STS($v$)s. During our investigations we discovered five further 4-chromatic STS(21)s, all obtained by applying smallish trades to Haddad's system. These are listed below together with the trades which produce them. They are easily seen to be pairwise non-isomorphic.

**STS(21) #1** (Haddad, [39])

$\{0,1,2\}$, $\{0,3,9\}$, $\{0,4,10\}$, $\{0,5,11\}$, $\{0,6,18\}$, $\{0,7,19\}$, $\{0,8,20\}$, $\{0,12,15\}$, $\{0,13,16\}$, $\{0,14,17\}$, $\{1,3,10\}$, $\{1,4,11\}$, $\{1,5,9\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{1,12,16\}$, $\{1,13,17\}$, $\{1,14,15\}$, $\{2,3,11\}$, $\{2,4,9\}$, $\{2,5,10\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,19\}$, $\{2,12,17\}$, $\{2,13,15\}$, $\{2,14,16\}$, $\{3,4,5\}$, $\{3,6,12\}$, $\{3,7,13\}$, $\{3,8,14\}$, $\{3,15,18\}$, $\{3,16,19\}$, $\{3,17,20\}$, $\{4,6,13\}$, $\{4,7,14\}$, $\{4,8,12\}$, $\{4,15,19\}$, $\{4,16,20\}$, $\{4,17,18\}$, $\{5,6,14\}$, $\{5,7,12\}$, $\{5,8,13\}$, $\{5,15,20\}$, $\{5,16,18\}$. $\{5,17,19\}$, $\{6,7,8\}$, $\{6,9,15\}$, $\{6,10,16\}$, $\{6,11,17\}$, $\{7,9,16\}$, $\{7,10,17\}$, $\{7,11,15\}$, $\{8,9,17\}$, $\{8,10,15\}$, $\{8,11,16\}$, $\{9,10,11\}$, $\{9,12,18\}$, $\{9,13,19\}$, $\{9,14,20\}$, $\{10,12,19\}$, $\{10,13,20\}$, $\{10,14,18\}$, $\{11,12,20\}$, $\{11,13,18\}$, $\{11,14,19\}$, $\{12,13,14\}$, $\{15,16,17\}$, $\{18,19,20\}$. The order of the full automorphism group is 108.

**STS(21) #2**

Replace the triples $\{1,6,19\}$, $\{1,12,16\}$, $\{6,10,16\}$, $\{10,12,19\}$ in system #1 by the triples $\{1,6,16\}$, $\{1,12,19\}$, $\{6,10,19\}$, $\{10,12,16\}$; a Pasch switch. The order of the full automorphism group is 12.

**STS(21) #3**

Replace the triples $\{0,1,2\}$, $\{0,5,11\}$, $\{1,4,11\}$, $\{2,4,9\}$, $\{2,5,10\}$, $\{9,10,11\}$ in system #1 by the triples $\{0,1,11\}$, $\{0,2,5\}$, $\{1,2,4\}$, $\{2,9,10\}$, $\{4,9,11\}$, $\{5,10,11\}$; a 6 block trade from an STS(9). The order of the full automorphism group is 12.

**STS(21) #4**

Replace the triples $\{0,7,19\}$, $\{0,8,20\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{2,6,20\}$, $\{2,8,19\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system #1 by the triples $\{0,7,8\}$, $\{0,19,20\}$, $\{1,6,8\}$, $\{1,7,19\}$, $\{1,18,20\}$, $\{2,6,19\}$, $\{2,8,20\}$, $\{6,7,20\}$, $\{8,18,19\}$; a 9 block trade from an STS(9). The order of the full automorphism group is 4.

**STS(21) #5**

Replace the triples $\{0,1,2\}$, $\{0,6,18\}$, $\{0,7,19\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,20\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system #1 by the triples $\{0,1,19\}$, $\{0,2,6\}$, $\{0,7,18\}$, $\{1,2,20\}$, $\{1,6,7\}$, $\{2,7,8\}$, $\{2,18,19\}$, $\{6,8,19\}$, $\{6,18,20\}$, $\{7,19,20\}$; a 10 block trade from an STS(9). The order of the full automorphism group is 3.

**STS(21) #6**

Replace the triples $\{0,1,2\}$, $\{0,7,19\}$, $\{0,8,20\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,19\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system #1 by the triples $\{0,1,8\}$, $\{0,2,7\}$, $\{0,19,20\}$, $\{1,2,6\}$, $\{1,7,19\}$, $\{1,18,20\}$, $\{2,8,20\}$, $\{2,18,19\}$, $\{6,7,20\}$, $\{6,8,19\}$, $\{7,8,18\}$; an 11 block trade from an STS(9). The order of the full automorphism group is 18.

For each admissible $v$, define the set $K(v) = \{\chi$: there exists a $\chi$-chromatic STS$(v)\}$. We know that $K(3) = \{2\}$ and that $K(7) = K(9) = K(13) = K(15) = \{3\}$, $\{3\} \subseteq K(19)$ and $K(21) = \{3,4\}$, the latter being one of the results of Chapter 4. Although we were unable to determine $K(19)$ precisely we were able to prove a set inclusion result.

Recall that an *independent set* is a subset of $V$, of an STS$(v)$, $(V,B)$, which does not contain a block, and that an independent set is *maximum* if it has the largest cardinality of any independent set in the design. Define $I(v) = \{\beta$: there exists an

STS($v$) with maximum independent set of cardinality $\beta\}$. In [7] it is shown that $I(19) = \{7, 8, 9, 10\}$ and $\{8, 9, 10\} \subseteq I(21)$. However, it is now known that in fact $I(21) = \{8, 9, 10\}$, which leads to an easy proof that every STS(21) is 4-colourable [24]; this is given in Chapter 4 (Theorems 4.4.1 and 4.4.2).

**Theorem 2.4.3** $\{3\} \subseteq K(19) \subseteq \{3, 4\}$.

**Proof.** It suffices to prove that every STS(19) is 4-colourable.

From the result about $I(19)$ quoted above, every STS(19) has an independent set of cardinality 7. Choose this set to be one of the colour classes, say $R$ (red). This gives $\binom{7}{2} = 21$ blocks with two red points and one non-red point and $(7\cdot 9) - (21\cdot 2) = 21$ blocks with one red point and two non-red points. This leaves 12 non-red points and 15 blocks of non-red points.

Now suppose that there exists a point $x$ which is contained in 5 red-free blocks, say $\{x, a, b\}$, $\{x, c, d\}$, $\{x, e, f\}$, $\{x, g, h\}$, $\{x, i, j\}$ where in addition we can assume that the triples of the STS(19) which contain the pairs $\{a, c\}$ or $\{a, d\}$ do not also contain the points $e$ or $f$. We now assign the points $a, c, d, e, f$ to a colour class $Y$ (yellow), the points $g, h, i, j$ to a colour class $B$ (blue) and the points $x, b, y$ to a colour class $G$ (green), where $y$ is the remaining non-red point. It is readily verified that there is no monochromatic block in the colour classes $Y$, $B$ or $G$.

If no such point $x$ exists then since $(15 \cdot 3)/12 > 3$, there exists a point $z$ which is contained in precisely 4 red-free blocks, say $\{z, a, b\}$, $\{z, c, d\}$, $\{z, e, f\}$, $\{z, g, h\}$. Let $i, j, k$ be the other 3 non-red points. Further, we can assume that the blocks of the STS(19) which contain the pairs $\{a, i\}$ or $\{b, i\}$ do not contain the points $c$ or $d$. Assign the points $a, b, c, d, i$ to a colour class $Y$ (yellow), the points $e, f, g, h$ to a colour class $B$ (blue) and the points $z, j, k$ to a colour class $G$ (green). Again it is readily verified that there is no monochromatic triple in the colour classes $Y$, $B$ or $G$. $\square$

## 2.5 Netto systems

The *Netto* systems belong a class of Steiner triple systems which admit a group of automorphisms acting doubly homogeneously but not doubly transitively on the points. They exist for orders $v = q = p^n$ where $p \equiv 7 \pmod{12}$ is prime and $n$ is odd. An elegant construction of them is described in [18]. We make use of the fact that for $v$ prime such systems are cyclic.

The system N(19) is 3-chromatic but only with colouring patterns $(c_1, c_2, c_3) = (7, 6, 6)$ or $(7, 7, 5)$; in fact it is one of only two STS(19)s we have discovered which is 3-chromatic with just these two patterns. If we represent N(19) on the set $V = \{0, 1, 2, \ldots, 18\}$ with blocks $B$ obtained by the action of the cyclic group generated by the mapping $i \mapsto i+1 \pmod{19}$ on the three starter blocks $\{0, 1, 8\}$, $\{0, 2, 5\}$ and $\{0, 4, 13\}$, then a 3-colouring with pattern $(c_1, c_2, c_3) = (7, 6, 6)$ is given by $R = \{0,$ $1, 2, 3, 4, 12, 14\}$, $Y = \{5, 7, 8, 11, 17, 18\}$, $B = \{6, 9, 10, 13, 15, 16\}$. A 3-colouring with pattern $(c_1, c_2, c_3) = (7, 7, 5)$ is given by $R = \{0, 1, 2, 3, 4, 12, 14\}$, $Y = \{6, 7,$ $9, 10, 11, 15, 16\}$, $B = \{5, 8, 13, 17, 18\}$.

The other STS(19) which is 3-chromatic only with colouring patterns $(c_1, c_2, c_3) = (7, 6, 6)$ or $(7, 7, 5)$ is also cyclic, this time with starter blocks $\{0, 1, 4\}$, $\{0, 2, 12\}$ and $\{0, 5, 13\}$. It is system A2 in the listing given in [53]. A 3-colouring with pattern $(c_1, c_2, c_3) = (7, 6, 6)$ is given by $R = \{0, 1, 2, 3, 7, 8, 17\}$, $Y = \{4,$ $5, 9, 13, 14, 15\}$, $B = \{6, 10, 11, 12, 16, 18\}$. A 3-colouring with pattern $(c_1, c_2, c_3) = (7, 7, 5)$ is given by $R = \{0, 1, 2, 3, 7, 8, 17\}$, $Y = \{4, 5, 9, 13,$ $14, 15, 16\}$, $B = \{6, 10, 11, 12, 18\}$.

There are four non-isomorphic, cyclic Steiner triple systems of order 19 and for completeness we give the 3-colouring patterns which are achievable for the remaining two systems. For system A1 with the three starter blocks $\{0, 1, 4\}$, $\{0, 2, 9\}$ and $\{0, 5, 11\}$, only 3-colourings with patterns $(c_1, c_2, c_3) = (7, 6, 6), (7, 7, 5)$ or $(8, 6, 5)$ can be achieved. Examples of these are $R = \{0, 1, 2, 3, 7, 8, 12\}$, $Y = \{4, 5, 9,$ $14, 17, 18\}$, $B = \{6, 10, 11, 13, 15, 16\}$; $R = \{0, 1, 2, 3, 7, 11, 12\}$, $Y = \{4, 8, 9,$ $13, 16, 17, 18\}$, $B = \{5, 6, 10, 14, 15\}$; $R = \{0, 1, 2, 3, 8, 11, 12, 17\}$, $Y = \{4, 5,$ $7, 13, 15, 16\}$, $B = \{6, 9, 10, 14, 18\}$ respectively. For system A3 with the three starter blocks $\{0, 1, 8\}$, $\{0, 2, 5\}$ and $\{0, 4, 10\}$, again only 3-colourings with patterns

$(c_1, c_2, c_3) = (7, 6, 6), (7, 7, 5)$ or $(8, 6, 5)$ can be achieved. Examples of these are $R = \{0, 1, 2, 3, 4, 12, 14\}$, $Y = \{5, 8, 10, 11, 15, 17\}$, $B = \{6, 7, 9, 13, 16, 18\}$; $R = \{0, 1, 2, 3, 4, 12, 15\}$, $Y = \{7, 8, 13, 14, 16, 17, 18\}$, $B = \{5, 6, 9, 10, 11\}$; $R = \{0, 1, 2, 4, 6, 11, 14, 17\}$, $Y = \{3, 7, 10, 15, 16, 18\}$, $B = \{5, 8, 9, 12, 13\}$ respectively.

The system N(31) only has a 3-colouring with pattern $(11, 10, 10)$. If we represent N(31) on the set $V = \{0, 1, 2, \ldots, 30\}$ with blocks $B$ obtained by the action of the group generated by the mapping $i \mapsto i + 1 \pmod{31}$ on the five starter blocks $\{0, 1, 6\}$, $\{0, 2, 12\}$, $\{0, 3, 16\}$, $\{0, 4, 24\}$ and $\{0, 8, 17\}$, then a 3-colouring with pattern $(c_1, c_2, c_3) = (11, 10, 10)$ is given by $R = \{0, 1, 2, 3, 4, 11, 18, 19, 23, 27, 29\}$, $Y = \{5, 6, 8, 12, 15, 17, 20, 25, 26, 30\}$, $B = \{7, 9, 10, 13, 14, 16, 21, 22, 24, 28\}$.

The system N(43) is 4-chromatic. If we represent N(43) on the set $V = \{0, 1, 2, \ldots, 42\}$ with blocks $B$ obtained by the action of the cyclic group generated by the mapping $i \mapsto i + 1 \pmod{43}$ on the seven starter blocks $\{0, 1, 37\}$, $\{0, 2, 14\}$, $\{0, 3, 21\}$, $\{0, 4, 19\}$, $\{0, 5, 35\}$, $\{0, 9, 32\}$ and $\{0, 10, 26\}$, then a 4-colouring is given by $R = \{16, 17, 18, 19, 23, 25, 28, 29, 32, 33, 34, 36, 40, 42\}$, $Y = \{3, 9, 14, 15, 21, 24, 27, 30, 35, 37, 38, 39, 41\}$, $B = \{4, 5, 6, 7, 13, 20, 22, 26, 31\}$, $G = \{0, 1, 2, 8, 10, 11, 12\}$.

## 2.6 Concluding remarks

This chapter has been concerned primarily with 3-chromatic Steiner triple systems and especially with the cardinalities of the colour classes. A 3-colouring may be either equitable or non-equitable. Thus 3-chromatic STS($v$)s can be partitioned into three types; (i) those which have both equitable and non-equitable 3-colourings, (ii) those which have only equitable 3-colourings, which are called *3-balanced*, and (iii) those which have only non-equitable 3-colourings, which by analogy are called *3-unbalanced.*

The spectrum of the first type is easy to determine; it is all admissible $v \geq 7$. For $v \equiv 3 \pmod 6$ use the Bose construction, [5], also given on page 25 of [16]. Let $Q = (S, \otimes)$ be a commutative idempotent quasigroup of odd order. Let the points of the Steiner triple system $V = S \times \{0,1,2\}$. The blocks are given by

$$\begin{aligned}&\{\{x_0, x_1, x_2\} : x \in S\}\\ &\cup \{\{x_i, y_i, (x \otimes y)_{i+1 \bmod 3}\} : x, y \in S, x < y, i = 0,1,2\}.\end{aligned}$$

If we use the subscripts to determine the colour classes then an equitable 3-colouring of the system is obtained. A non-equitable colouring may be obtained by recolouring any one of the points with subscript 0 as the colour of the class with subscript 1.

For $v \equiv 1 \pmod 6$ use the Skolem construction, [67], also given on page 26 of [16]. Let $Q = (S, \otimes)$ be a commutative quasigroup of even order where precisely half of the symbols satisfy $x \otimes x = x$. Let the points of the Steiner triple system be $V = (S \times \{0,1,2\}) \cup \{\infty\}$. Then the block set is

$$\begin{aligned}&\{\{x_0, x_1, x_2\} : x \in S, x \otimes x = x\}\\ &\cup \{\{\infty, x_i, (x \otimes x)_{i+1 \bmod 3}\} : x \in S, x \otimes x \neq x, i = 0,1,2\}\\ &\cup \{\{x_i, y_i, (x \otimes y)_{i+1 \bmod 3}\} : x, y \in S, x < y, i = 0,1,2\}.\end{aligned}$$

Again, an equitable 3-colouring is determined by the subscripts, together with $\infty$ assigned to the colour with subscript 0. A non-equitable colouring may then be obtained by recolouring any one of the points with subscript 1 as the colour with subscript 2.

The existence of 3-balanced STS($v$)s was considered by Colbourn, Haddad and Linek in [12], where they obtain the following result.

**Theorem 2.6.1** *With the possible exceptions of $v \in \{19, 21, 37, 49, 55, 57, 67, 69, 85, 109, 139\}$, for all admissible $v \geq 15$, there exists a 3-balanced STS($v$).*

**Proof.** See the original paper. □

However, we are able to dispose of all the exceptions in Theorem 2.6.1 except $v = 19$. For $v \geq 25$, we refer the reader to Chapter 3 of this thesis and for completeness we give below two (non-isomorphic) examples of 3-balanced STS(21)s. In neither case is the 3-colouring unique in the sense of Chapter 3.

**STS(21) #1**

`5ackfejidh 6bajfkegi7 cbkgfhj8di fhgk9edihj`

`afejikbgck jchgkdihej fkjgkhiijk`

A 3-colouring is given by $R = \{0, 1, 2, 3, 8, \mathrm{e}, \mathrm{i}\}$, $Y = \{4, 6, 7, \mathrm{c}, \mathrm{f}, \mathrm{j}, \mathrm{k}\}$, $B = \{5, 9, \mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{g}, \mathrm{h}\}$. The system is a Kirkman triple system. The number of blocks of each colour type is $RRY = 11$, $RRB = 10$, $RYY = 10$, $RBB = 11$, $YYB = 11$, $YBB = 10$, $RYB = 7$. The order of the full automorphism group is 42.

**STS(21) #2**

`4a9bijkfgh 5ajkibghfb 9kighfj5cd eijkdecbjk`

`iecdakij8f ghghfhfjij kjkikigehk`

A 3-colouring is given by $R = \{0, 1, 2, 6, 9, \mathrm{e}, \mathrm{g}\}$, $Y = \{3, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{f}, \mathrm{h}, \mathrm{j}\}$, $B = \{4, 5, 7, 8, \mathrm{a}, \mathrm{i}, \mathrm{k}\}$. The number of blocks of each colour type is $RRY = 10$, $RRB = 11$, $RYY = 11$, $RBB = 10$, $YYB = 10$, $YBB = 11$, $RYB = 7$. The order of the full automorphism group is 2.

In [26] we offered the STS(19) case as a problem.

**Problem 1** *Does there exist a 3-balanced STS(19)?*

In section 3 of [26], we exhibited the first published examples of 3-unbalanced STS($v$); for $v = 25$, 27, 31, 33, 37 and 39. It was previously known that a 3-unbalanced STS($v$)s do not exist for $v = 7$, 9, 13 and 15 and we succeeded in proving that none exist for $v = 19$. Also we conjectured that 3-unbalanced STS($v$)s exist for all admissible $v \geq 25$, and this interesting problem is resolved in Chapter 3 of this thesis (Theorem 3.1.2). The case of STS(21) was offered as our second problem in [26].

**Problem 2** *Does there exist a 3-unbalanced STS(21)?*

Finally, we still seek an answer to the following, also posed in [26].

**Problem 3** *Does there exist a 4-chromatic STS(19)?*

In view of the enumeration of the STS(19)s by Kaski and Östergård [43], it should at present be computationally feasible to resolve Problems 1 and 3, and we expect this to be achieved in the not-too-distant future. On the other hand, in the few years that have elapsed since the preparation of [26] we are still no further towards solving Problem 2.

# Chapter 3

# Uniquely 3-colourable Steiner triple systems

## 3.1 Introduction

Let $S = (V, \mathcal{B})$ be an STS$(v)$. Recall that a *3-colouring* of $S$ is a surjection $\phi : V \to C$, where $C$ is a set of cardinality 3, whose elements are called *colours*, such that $|\phi(T)| > 1$ for all $T \in \mathcal{B}$; i.e. every block contains at least two points with different colours. Throughout this chapter we use a fixed set of colours $C = \{R, Y, B\}$, and if we want to refer to the colour class sizes $c_1 = |\phi^{-1}(R)|$, $c_2 = |\phi^{-1}(Y)|$ and $c_3 = |\phi^{-1}(B)|$ of $\phi : V \to \{R, Y, B\}$, we describe $\phi$ as a $(c_1, c_2, c_3)$ 3-colouring. We usually assign the colours $R$, $Y$ and $B$ so that $c_1 \geq c_2 \geq c_3$.

Let $S$ be an STS$(v)$ and suppose it has a 3-colouring, $\phi$, with colours $\{R, Y, B\}$ such that $|\phi^{-1}(R)| \geq |\phi^{-1}(Y)| \geq |\phi^{-1}(B)|$. We say that the 3-colouring $\phi$ is *equitable* if $|\phi^{-1}(R)| - |\phi^{-1}(B)| \leq 1$, otherwise $\phi$ is *non-equitable.* If every 3-colouring of $S$ is equitable, $S$ is called *3-balanced.* If every 3-colouring of $S$ can be obtained from $\phi$ by a permutation of the colours, we say that $S$ has a *unique* 3-colouring, or that $S$ is *uniquely 3-colourable.*

It is worth emphasizing that our definition of unique 3-colouring must be thought of as applying to labelled objects. Some authors weaken the definition such that uniqueness implies only that all colourings of a labelled system yield the same (up to isomorphism) colouring of the corresponding unlabelled system. For example, under our definition the projective STS(15), number #1 in the list of [16], has precisely 2016 3-colourings (all with colour class sizes $\{5, 5, 5\}$), whilst under the weaker definition, the 3-colouring of this system is unique [30]. The two definitions are equivalent for Steiner triple systems which have trivial full automorphism group.

We are interested in two special properties which we now define.

**Definition 3.1.1** A *type-I* STS($v$) is a 3-balanced STS($v$) with a unique (necessarily equitable) 3-colouring.

**Definition 3.1.2** A *type-II* STS($v$) is an STS($v$) with a unique 3-colouring, and that 3-colouring is non-equitable.

The cyclic STS(33) with starter blocks $\{0,1,3\}$, $\{0,4,10\}$, $\{0,5,18\}$, $\{0,7,19\}$, $\{0,8,17\}$, $\{0,11,22\}$ is known to be of type-I (Colbourn, Haddad & Linek [12]), and, more recently, Forbes, Grannell & Griggs [26] have shown that type-I STS($v$)s exist also for $v = 25$, 27, 31, 37, 49, 55 and 57. Leaving aside uniqueness, Colbourn, Haddad & Linek [12] also proved that a 3-balanced STS($v$) exists for every admissible $v > 15$, except possibly for $v \in \{19, 21, 37, 49, 55, 57, 67, 69, 85, 109, 139\}$. Although the exceptional cases, apart from 19, have been dealt with in [26], we think it is of some interest to extend the result to STS($v$)s of type-I, i.e. uniquely 3-colourable as well as 3-balanced. So for the first theorem of this chapter, proved in section 3.3, we have:

**Theorem 3.1.1** *A type-I STS($v$) exists for every admissible $v \geq 25$.*

Since it is known that no type-I STS($v$) exists for any $v \leq 15$, the only cases left undecided by Theorem 3.1.1 are $v = 19$ and $v = 21$.

The STS(7) and STS(9) each have an equitable 3-colouring, and Mathon, Phelps & Rosa [53] showed that the same is true for the 80 STS(15)s. On page 343 of Colbourn & Rosa [16], the question was posed as to whether every 3-colourable STS($v$) has an equitable 3-colouring. For $v = 19$ this is indeed the case [26]. Furthermore, Haddad & Rödl [40] showed that if there exists a 3-colouring of an STS($v$), the colour class sizes, $c_1$, $c_2$, $c_3$, say, satisfy

$$(c_1 - c_2)^2 + (c_1 - c_3)^2 + (c_2 - c_3)^2 \leq 2v,$$

and hence for large $v$ they cannot deviate by too much from equality.

The question of the existence of 3-colourable STS($v$)s which have only a non-equitable 3-colouring was resolved affirmatively in [26] by exhibiting type-II STS($v$)s for $v \in \{25, 27, 31, 33, 39\}$. However these were isolated examples. In the next section we describe some constructions that allow us to extend the known spectrum of type-II STS($v$)s to all admissible $v$ with one exception. We now state our second theorem, which we prove in section 3.4.

**Theorem 3.1.2** *A type-II STS($v$) exists for every admissible $v \geq 25$.*

Since we already know that there is no type-II STS($v$) for any $v \leq 19$, the only case that remains in doubt is $v = 21$.

## 3.2 Preliminary lemmas

Let $s \geq 3$ and $g_1, g_2, \ldots, g_s$ be positive integers. A type $(g_1, g_2, \ldots, g_s)$ $\{3\}$-GDD (group divisible design) is a triple $(V, \mathcal{G}, \mathcal{B})$, where $V$ is a set of *points*, $\mathcal{G}$ is a collection of $s$ sets, called *groups*, $G_1$, $G_2, \ldots, G_s$ of cardinality $g_1, g_2, \ldots, g_s$, respectively, whose elements are taken from $V$, and $\mathcal{B}$ is a set of 3-element subsets of $V$, called *blocks*, such that every pair of elements $x, y \in V$, appears in exactly one block of $\mathcal{B}$ if and only if $x$ and $y$ belong to different groups. If $x$ and $y$ belong to the same group, no block contains $\{x, y\}$. In an alternative notation we gather together equal group sizes and count them using superscripts; thus, for example, the type specification (24, 24, 24, 30) would be denoted by $24^3 30^1$.

By analogy with Steiner triple systems, a 3-colouring of a type $(g_1, g_2, \ldots, g_s)$ $\{3\}$-GDD, $(V, \{G_1, G_2, \ldots, G_s\}, \mathcal{B})$, is a function $\phi : V \to \{R, Y, B\}$ such that for any $T \in \mathcal{B}$, $\phi(T) > 1$. We also define the $3 \times s$ *3-colouring matrix* $(h_{i,j})$ where $h_{1,j}, h_{2,j}, h_{3,j}$ are the numbers of points in $G_j$ which are coloured $R$, $Y$, $B$, respectively.

**Lemma 3.2.1** (Colbourn, Hoffman & Rees [13]) *Let $g$, $t$ and $u$ be non-negative integers. Then there exists a $\{3\}-GDD$ of type $g^t u^1$ if and only if all the following conditions are satisfied:*

1. *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*

2. *$u \leq g(t-1)$ or $gt = 0$;*

*3.* $g(t-1)+u \equiv 0 \pmod 2$ *or* $gt=0$*;*

*4.* $gt \equiv 0 \pmod 2$ *or* $u=0$*;*

*5.* $\frac{1}{2}g^2t(t-1)+gtu \equiv 0 \pmod 3$*.*

**Proof.** See the original paper. □

The next lemma is a variation of Lemma 3.1 in Colbourn, Haddad & Linek [12].

**Lemma 3.2.2** *Let* $u_1 \geq u_2 \geq u_3$ *be non-negative integers with* $u = u_1+u_2+u_3 \equiv 1$ *or* $3 \pmod 6$ *and let* $s \geq 3$*. Suppose there exists a* $\{3\}$*−GDD of type* $(g_1, g_2, \ldots, g_s)$ *and for each* $j$*,* $j=1,2,\ldots,s$*, an STS*$(3g_j+u)$*,* $S_j$*, with a sub-STS*$(u)$*. Suppose also that each* $S_j$ *has a unique 3-colouring comprising* $g_j+u_1$*,* $g_j+u_2$ *and* $g_j+u_3$ *points of colour R, Y and B, respectively, which induces a* $(u_1,u_2,u_3)$ *3-colouring on the sub-STS*$(u)$*.*

*Let* $g = g_1+g_2+\cdots+g_s$*. Then there exists an STS*$(3g+u)$ *which has a unique 3-colouring consisting of* $g+u_1$*,* $g+u_2$ *and* $g+u_3$ *points of colour R, Y and B, respectively. Moreover, this 3-colouring induces a* $(u_1,u_2,u_3)$ *3-colouring on a sub-STS*$(u)$*.*

**Proof.** Given a type $(g_1, g_2, \ldots, g_s)$ $\{3\}$-GDD, $(V', \mathcal{G}', \mathcal{B}')$, use Wilson's fundamental construction (see, for example, Colbourn & Rosa [16], Theorem 3.3) to create a 3-colourable $(3g_1, 3g_2, \ldots, 3g_s)$ $\{3\}$-GDD, $(V, \mathcal{G}, \mathcal{B})$, where in each group the colour classes are of equal cardinality. Let $V$ be the set of points $V' \times \{0,1,2\}$. For brevity, here and elsewhere, we write $x_i$ instead of $(x,i)$. If the groups in $\mathcal{G}'$ are $G'_1$, $G'_2$, ..., $G'_s$, let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$, where $G_j = G'_j \times \{0,1,2\}$, $j=1,2,\ldots,s$. For each block $\{x,y,z\} \in \mathcal{B}'$, place in $\mathcal{B}$ the nine blocks

$$\begin{gathered}\{x_0,y_0,z_1\},\quad \{x_1,y_1,z_2\},\quad \{x_2,y_2,z_0\},\\ \{x_0,y_1,z_0\},\quad \{x_1,y_2,z_1\},\quad \{x_2,y_0,z_2\},\\ \{x_0,y_2,z_2\},\quad \{x_1,y_0,z_0\},\quad \{x_2,y_1,z_1\}.\end{gathered} \tag{3.1}$$

For each $x$ in $V'$ assign colours $R$, $Y$, $B$ to $x_0$, $x_1$, $x_2$, respectively. It is clear that there are no monochrome blocks in $\mathcal{B}$ and that each group in $\mathcal{G}$ is equitably 3-coloured.

Now append a set $U$ of $u$ new points to $V$. For each $j = 1, 2, \ldots, s$ there is a unique 3-colouring of $S_j$ with $g_j + u_1$, $g_j + u_2$ and $g_j + u_3$ points of colour $R$, $Y$ and $B$, respectively. Label the points of $S_j$ such that (i) the points of the sub-STS($u$) of $S_j$ receive the same labels as the points of $U$, with $u_1$, $u_2$ and $u_3$ points of colour $R$, $Y$ and $B$, respectively; (ii) the other $3g_j$ points of $S_j$ receive the same labels as the points of $G_j$, with $g_j$ points of each colour. The blocks, $\mathcal{B}_j$ of the relabelled $S_j$ are added to $\mathcal{B}$ except that if a block belongs to the sub-STS($u$) it is added only once.

Thus $S = (V \cup U, \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_s)$ is an STS$(3g + u)$ with $s$ subsystems, STS$(3g_j + u)$, $j = 1, 2, \ldots, s$, each having the common sub-STS($u$). Furthermore, $S$ has a $(g + u_1, g + u_2, g + u_3)$ 3-colouring arising from the equitable 3-colouring of the underlying type $(3g_1, 3g_2, \ldots, 3g_s)$ $\{3\}$-GDD and the $(u_1, u_2, u_3)$ 3-colouring of the STS($u$).

Since the 3-colouring of each $S_j$ is unique, any other 3-colouring of $S$ must arise from a permutation of the colours of some of the $S_j$. However, if $u_2 \geq 1$, a permutation, $\pi$, of the colours of $S_p$, say, induces the same permutation of the colours of $U$ and that in turn induces the same permutation of the colours of all the other $S_j$. The result is that $\pi$ acts on the colours of the whole of $S$. Hence the 3-colouring of $S$ is unique.

Now suppose $u_2 = u_3 = 0$. Then $|U| = u_1 = 1$ and, by a similar argument, changing the colour $R$ to either $Y$ or $B$ in one of the $S_j$ causes the same change in the single point of $U$ (which is originally coloured $R$) and hence in the whole of $S$. So we can assume that any new 3-colouring of $S$ does not affect points originally coloured $R$.

We examine what happens if the colours $Y$ and $B$ are swapped in some but not all of the $S_j$. We can assume (by relabelling the groups if necessary) that the colours $Y$ and $B$ are swapped in $S_1$ but not in $S_2$.

Suppose $x \in G'_1$, $y \in G'_2$ and $\{x, y, z\} \in \mathcal{B}'$. By the method of construction, $S$ contains the nine blocks given in (3.1) and the original 3-colouring maps $x_0, y_0, z_0$

to $R$, $x_1, y_1, z_1$ to $Y$ and $x_2, y_2, z_2$ to $B$. After the swap, $\{x_1, y_2\}$ are coloured $B$ and $\{x_2, y_1\}$ are coloured $Y$. But then whatever colour is assigned to $z_1$ there will be a monochrome block somewhere in (3.1). □

**Lemma 3.2.3** *In an equitable 3-colouring of an STS$(6k+3)$ there is a block containing one point of each colour.*

**Proof.** The number of such blocks is $(2k+1)(3k+1) - 3(2k+1)k = 2k+1$. □

**Lemma 3.2.4** *In a 3-colouring of an STS$(6k+3)$ with $2k+2$ points of colour $R$, $2k+1$ points of colour $Y$ and $2k$ points of colour $B$, there is at least one block containing two points of colour $R$ and one point of colour $Y$.*

**Proof.** There are $(k+1)(2k+1)$ blocks containing two points of colour $R$. At least one of these blocks must contain a point of colour $Y$ since there are only $4k(k+1)$ $\{R, B\}$ pairs. □

**Lemma 3.2.5** (Colbourn, Haddad & Linek [12]) *If $g$ is a positive integer, there exists a type $(3g+2)^3$ $\{3\} - GDD$, $(V, \{G_1, G_2, G_3\}, \mathcal{B})$, with 3-colouring matrix*

$$\begin{pmatrix} g+1 & g+1 & g+1 \\ g+1 & g+1 & g \\ g & g & g+1 \end{pmatrix}.$$

**Proof.** This is Lemma 2.4 of [12]. □

**Lemma 3.2.6** *If $g$ is a positive integer, there exists a type $(3g+1)^3$ $\{3\} - GDD$, $(V, \{G_1, G_2, G_3\}, \mathcal{B})$, with 3-colouring matrix*

$$\begin{pmatrix} g+1 & g+1 & g \\ g & g & g+1 \\ g & g & g \end{pmatrix}.$$

**Proof.** We construct a design with the desired property. Put $V = \{0, 1, \ldots, 3g\} \times \{1, 2, 3\}$, $G_j = \{0, 1, \ldots, 3g\} \times \{j\}$, $j = 1, 2, 3$ and

$$\mathcal{B} = \{\{x_1, y_2, z_3\} : 0 \leq x, y, z \leq 3g, z \equiv x + y \pmod{3g+1}\}.$$

Denoting $\{a_i, b_i, \ldots, z_i\}$ by $\{a, b, \ldots, z\}_i$, one can then verify that a valid 3-colouring is given by the matrix

$$\begin{pmatrix} \{2g, 2g+1, \ldots, 3g\}_1 & \{0, 1, \ldots, g\}_2 & \{g, g+1, \ldots, 2g-1\}_3 \\ \{g, g+1, \ldots, 2g-1\}_1 & \{2g+1, 2g+2, \ldots, 3g\}_2 & \{2g, 2g+1, \ldots, 3g\}_3 \\ \{0, 1, \ldots, g-1\}_1 & \{g+1, g+2, \ldots, 2g\}_2 & \{0, 1, \ldots, g-1\}_3 \end{pmatrix}.$$

□

**Lemma 3.2.7** *Let $g$ and $w$ be positive integers with $w$ even and $w \geq 2$. Let $u = 3w + 1$. Suppose there exists a type-II $STS(3g + 2 + u)$ in which a $(g + w + 2$, $g+w+1$, $g+w)$ 3-colouring induces a $(w+1$, $w$, $w)$ 3-colouring on a sub-$STS(u)$. Then there exists a type-II $STS(9g + 6 + u)$.*

**Proof.** Start with the $\{3\}$-GDD of Lemma 3.2.5 and a set $U$ of $u$ points with $w + 1$ coloured $R$ and $w$ each coloured $Y$ and $B$. For $j = 1, 2$, let $\mathcal{B}_j$ be the blocks of the given STS$(3g + 2 + u)$ with its $(g + w + 2, g + w + 1, g + w)$ 3-colouring labelled such that the points of the sub-STS$(u)$ receive the same labels and colours as $U$ and the remaining points receive the same labels and colours as $G_j$. Let $\mathcal{B}_3$ be the set of blocks constructed in a similar manner using $G_3$ and the same 3-coloured STS$(3g + 2 + u)$ but with the colours $Y$ and $B$ interchanged.

Then $(V \cup U, \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is an STS$(9g + 6 + u)$ which has a $(3g + w + 4$, $3g+w+2$, $3g+w+1)$ 3-colouring. The 3-colouring is unique by the same argument as that used in the proof of Lemma 3.2.2. □

**Lemma 3.2.8** *Let $g$ and $w$ be positive integers with $w$ odd. Let $u = 3w$. Suppose there exists a type-I $STS(3g + 1 + u)$ in which a $(g + w + 1$, $g + w$, $g + w)$ 3-colouring induces an equitable 3-colouring on a sub-$STS(u)$. Then there exists a type-II $STS(9g + 3 + u)$.*

**Proof.** This time we use the $\{3\}$-GDD of Lemma 3.2.6 and a set $U$ of $u$ points with $w$ points each coloured $R$, $Y$ and $B$. For $j = 1, 2$, let $\mathcal{B}_j$ be the blocks of the given 3-coloured STS$(3g + 1 + u)$ labelled such that the points of the sub-STS$(u)$ receive the same labels and colours as $U$ and the remaining points receive the same labels and colours as $G_j$. Let $\mathcal{B}_3$ be the set of blocks constructed in a similar manner from $G_3$ and the STS$(3g + 1 + u)$ with the colours $R$ and $Y$ interchanged.

Then $(V \cup U, \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is an STS$(9g+3+u)$ which has a $(3g+w+2,$ $3g+w+1,$ $3g+w)$ 3-colouring. Again, uniqueness follows as in the proof of Lemma 3.2.2. $\square$

## 3.3 Proof of Theorem 3.1.1

First we deal with $v \equiv 3 \pmod 6$.

We use Lemma 3.2.2 with $u = 3$ and $u_1 = u_2 = u_3 = 1$ together with a finite set of *starter systems.* If for $t \geq t_0$ there exists a type $8^{3t+r}h^1$ $\{3\}$-GDD and a type-I STS(27) together with, if $h \neq 0$, a type-I STS$(3h+3)$, it follows from Lemma 3.2.2 and Lemma 3.2.3 that there exists a type-I STS$(72t+24r+3h+3)$ for all $t \geq t_0$.

There are eight starter systems, type-I STS$(v)$s for $v$ = 27, 33, 39, 45, 51, 57, 69 and 87. The first six systems are presented in section 3.5. For the last two, see Table 3.3 (page 47).

In Table 3.1 (page 46) we give the details for the construction of type-I STS$(v)$s for each admissible residue class modulo 72. Lemma 3.2.1 ensures the existence of the $\{3\}$-GDDs in column 1 except for those which generate the entries in column 4. The exceptions which are not listed in section 3.5 are constructed separately by Lemma 3.2.2 and possibly Lemma 3.2.3 using different $\{3\}$-GDDs, as in Table 3.3 (page 47). For this purpose, observe that the STS(27), STS(33) and STS(39) all have suitable subsystems of the appropriate orders.

The procedure for $v \equiv 1 \pmod 6$ is similar. We construct type-I STS$(v)$s for each admissible residue class modulo 54 using Lemma 3.2.2 with $u = 7$, $u_1 = 3$ and $u_2 = u_3 = 2$. See Table 3.2 (page 46). There are three starter systems, each with a suitable sub-STS(7): type-I STS$(v)$s for $v$ = 25, 31 and 37, listed in section 3.6. Systems for $v = 67$ and 73 are constructed as indicated in Table 3.3 (page 47).

The proof is completed by exhibiting in section 3.6 specific examples of type-I STS$(v)$s for $v$ = 43, 49 and 55 as well as an additional STS(31), with a suitable sub-STS(13), for use in the construction of the STS(67).

## 3.4 Proof of Theorem 3.1.2

For $v \equiv 3 \pmod 6$ we use Lemma 3.2.2 with $u = 3$, $u_1 = 2$, $u_2 = 1$ and $u_3 = 0$ together with a set of starter systems, STS($v$) for $v = 27$, 33, 39, 45, 51, 129, 141, and 159. Lemma 3.2.4 ensures that each system has a suitable sub-STS(3). Table 3.4 (page 47) gives the details for the construction of a type-II STS for each admissible residue class modulo 72.

Table 3.6 (page 48) deals with all the the exceptions apart from $v = 57$. The systems in column 3 for $v = 27$, 33, 39, 45 and 51 are presented in section 3.7.

When $v \equiv 1 \pmod 6$ we construct type-II STS($v$)s for each admissible residue class modulo 54 using Lemma 3.2.2 with $u = 7$, $u_1 = u_2 = 3$ and $u_2 = 1$, as in Table 3.5 (page 47). There are three starter systems with suitable sub-STS(7), type-II STS($v$)s for $v = 25$, 31 and 37, listed in section 3.8. (It is fortunate that the one known example of an STS(25) which is only non-equitably 3-colourable happens to satisfy our requirements for a suitable starter system.) We also give an additional STS(31) (containing a sub-STS(13)) for dealing with the case $v = 67$ as well as examples of a type-II STS($v$) for $v = 43$ and 49.

The systems STS(27) #2 and STS(33) #3 of section 3.7 both satisfy the requirements for Lemma 3.2.7, with $w = 4$ and $u = 13$. Hence we can use the lemma to construct a type-II STS(55) ($g = 4$) and a type-II STS(73) ($g = 6$).

It is worth pointing out that, as with any STS($6k+1$) constructed by Lemma 3.2.7 and unlike all other systems presented in this paper, the STS(55) and the STS(73) have the property that there is a spread of three in the colour class sizes of their unique 3-colourings.

As an aside, we would like to speculate that arbitrary deviations from the 3-balanced property occur for all sufficiently large orders. Indeed, as a first step in this direction, one might try to use Lemmas 3.2.2, 3.2.7 and 3.2.8 to prove that *for any positive integer $d$ there exists an STS($v$) with a unique 3-colouring having colour class sizes $(c_1, c_2, c_1 - d)$, where $c_1 \geq c_2 \geq c_1 - d$.*

Finally, we use Lemma 3.2.8 with $g = 5$ and $w = 3$ for constructing a type-II STS(57) from the 3-balanced STS(25) #2 (which has a suitable sub-STS(9)) of section 3.6.

## 3.5 Type-I systems with $v \equiv 3 \pmod 6$

In this section we give the details of type-I STS($v$)s for $v = 27$, 33, 39, 45, 51 and 57, which we require as starter systems in the proof of Theorem 3.1.1.

Here and in the next three sections we employ the compact notation described in [26] to indicate the blocks of a Steiner triple system.

Assume the point set of the STS($v$) is $\{0, 1, \ldots, v-1\}$. The system is then represented as a string of $b = \frac{1}{6}v(v-1)$ symbols, $s_1$, $s_2$, ..., $s_b$. Using the standard lexicographical order, symbol $s_i$ is the largest element $z_i$ in the $i^{\text{th}}$ block $\{x_i, y_i, z_i\}$, where $x_i \leq y_i \leq z_i$. The other two elements are easily recoverable from $z_i$: $(x_i, y_i)$ is the smallest pair that does not appear earlier in the lexicographical ordering of the blocks. Integers 10, 11, ..., 35 are represented by lower-case letters a, b, ..., z, integers 36, 37, ..., 56 by upper case letters A, B, ..., U.

STS(27)

`2687mnlqkp oji876lqko jmpin768pi oknqjml5jl qipknomnom`

`jilkqpqmpn ljiko8kpjm oilnqokilq nmpjijnpmo qlkbfhghgf`

`gfhehlqnpp qonopqq`

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(9, 9, 9)$ and is given by $R = \{$`0, 1, 5, 8, 9, a, k, o, p`$\}$, $Y = \{$`2, 4, b, c, f, h, i, n, q`$\}$, $B = \{$`3, 6, 7, d, e, g, j, l, m`$\}$. The number of blocks of each colour type is $RRY = 19$, $RRB = 17$, $RYY = 17$, $RBB = 19$, $YYB = 19$, $YBB = 17$, $RYB = 9$. The 3-colouring of the STS(27) induces an equitable 3-colouring on the sub-STS(9) $\{$`i, j, k, l, m, n, o, p, q`$\}$. The system has trivial full automorphism group.

STS(33)

`ed8bc9amwi lvqrtu3579 bdgtjrnsuq w689cekusq mopvwa7bde`

`rimtuwsqvc desqvkumwt readhvowts prucdlmnto wurvbethwr`

`mquvseulrn vstpwcipoq lnuwvjnkpv oswuorlsnp wtvptjlqrv`

`swwouvkpqt snkqrpuvwt vswuot`

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(11, 11, 11)$ and is given by $R = \{$`0, 2, 4, 9, c, i, j, n, r, s, t`$\}$, $Y = \{$`1, 5, 6, 8,` d, h, k, l, m, o, v}, $B$ = {3, 7, a, b, e, f, g, p, q, u, w}. The number of blocks of each colour type is $RRY = 26$, $RRB = 29$, $RYY = 29$, $RBB = 26$, $YYB = 26$, $YBB = 29$, $RYB = 11$. This 3-colouring induces an equitable 3-colouring on the sub-STS(15) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e}. The system has trivial full automorphism group.

STS(39)

```
85CBb9Avuw srtyxzpoq4 6BAaCuwvrt sxzyoqp7AB Cbwvutsrzy
xqpoA9bBsr tvuwpoqyxz 87bCrtsuwv oqpxzyaCAt srwvuqpozy
xCBpoqyxzv uwsrtaBoqp xzyuwvrtsA qpozyxwvut sryxzpoqsr
tvuwxzyoqp rtsuwvzyxq potsrwvukh CBnlAgiBAm CjABCnAlnB
kjnCmCACBm BAwtCBzxAs uBAyCvABCz AxzBwvzCyC ACByBAC
```

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(13, 13, 13)$ and is given by $R$ = {0, 3, 6, 9, c, f, i, l, o, r, u, x, A}, $Y$ = {1, 4, 7, a, d, g, j, m, p, s, v, y, B}, $B$ = {2, 5, 8, b, e, h, k, n, q, t, w, z, C}. The number of blocks of each colour type is $RRY = 63$, $RRB = 15$, $RYY = 15$, $RBB = 63$, $YYB = 63$, $YBB = 15$, $RYB = 13$. Furthermore, this 3-colouring of the STS(39) induces an equitable 3-colouring on each of three sub-STS(15)s: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, A, B, C}, {c, d, e, f, g, h, i, j, k, l, m, n, A, B, C} and {o, p, q, r, s, t, u, v, w, x, y, z, A, B, C}. The system has trivial full automorphism group.

STS(45)

```
ane89bdgwv FDEIGCABzx yH96c7eduv wDEFGHIABC xyz5d8beGv
wuEFDHIBCA yzxocbEdCA BzxywuvGHF DIbBeaAGCx yzuvwHIDEF
edcBCAyzxv wuHIGEFD9I dFDEjHCAGz xywuvaDEFG HIABCxyzuv
wcEFDHIGBC Ayzxvwunez xyCABFDwuv IGHmxyzABC DFuvwGHEey
zxBCAEvwuH IGFIHwuvzx yFDECABGHI uvwxyzDEFA BCHIGvwuyz
xEFDBCAprt noqsolrmts BksnqtrqIs qptGtsrosp FrHEqttEGI
CDFDAGBIHz HCFIDGFHFE IIHGDHEGII
```

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(15, 15, 15)$ and is given by $R$ = {0, 3, 6, 9, c, f, i, l, o, r, u, y, A, B, G}, $Y$ = {1, 4, 7, a, d, g, j, m, p, s, x, z, D, F, I}, $B$ = {2, 5, 8, b, e, h, k, n, q, t, v, w, C, E, H}. The number of blocks of each colour type is $RRY = 58$, $RRB = 47$, $RYY = 47$, $RBB = 58$, $YYB = 58$, $YBB = 47$, $RYB = 15$. The system has trivial full automorphism group.

STS(51)

```
qgrjsktlum vnwoxypONM LKJIHrjskt lumvnwoxph zAONMLKJIz
ktlumvnwox phqABCONML KJflumvnwo xphiBDtECO NMLKumvnwo
xphqrDCBGF EONMLvnwox phqirjDIHG FEONMwoxph qirjsEKJIH
GFONxphqir jskFMLKJIH GOhqirjskt GONMLKJIHi rjsktlHzON
MLKJIjsktl uIBAONMLKJ ktlumJDCBO NMLKlumvKF EDCONMLmvn
LHGFEDONMn wMJIHGFEON oLyNKJIHGF OqONMLKJIH GHzIAJBKCL
DMENFOGIAJ BKCMENFOGy JBKCLDMENF OGyHKCLDME NFOGyHzLDM
ENFOGyHzIM ENFOGyHzIA NFOGyHzIAJ OGyHzIAJBH zIAJBKLzIA
JBKCAJBKCL KCLDCLDMDM EENFD
```

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(17, 17, 17)$ and is given by $R = \{$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, D, G$\}$, $Y = \{$f, g, h, i, j, k, l, m, n, p, r, s, t, u, v, w, x$\}$, $B = \{$o, q, y, z, A, B, C, E, F, H, I, J, K, L, M, N, O$\}$. The number of blocks of each colour type is $RRY = 95$, $RRB = 41$, $RYY = 41$, $RBB = 95$, $YYB = 95$, $YBB = 41$, $RYB = 17$. The system has trivial full automorphism group.

STS(57)

```
3U9RdPqxwE FzutNCGKBL yAHJDOQT4a SeQryxFGAv uODHLCMzBI
KEPRU5bTfR szyGHBwvPE IMDNACJLFQ S6cUgStAzH ICxwQFJNEO
BDKMGRT7dh TuBAIJDyxR GKOFPCELNH SU8eiUvCBJ KEzySHLPGQ
DFMOIT9fjw DCKLFAzTIM QHREGNPJUa gkxEDLMGBA UJNRISFHOQ
KbhlyFEMNH CBKOSJTGIP RLcimzGFNO IDCLPTKUHJ QSMdjnAHGO
PJEDMQULIK RTNekoBIHP QKFENRMJLS UOflpCJIQR LGFOSNKMTP
```

gmqDKJRSMH GPTOLNUQhn rELKSTNIHQ UPMORiosFM LTUOJIRQNP
SjptGNMUPK JSROQTkquH ONQLKTSPRU lrvIPORMLU TQSmswJQPS
NMURTntxKR QTONSUouyL SRUPOTpvzM TSQPUqwANU TRQrxBOUSR
syCPTStzDQ UTuAERUvBF SwCGTxDHUy EIzFJAGKBH LCIMDJNEKO
FLPGMQHNRI OSJPTKQULR MSNTOUPQRS TU

The 3-colouring is unique, modulo permuting the colours, with colour class sizes $(19, 19, 19)$ and is given by $R = \{$0, 3, 6, 9, c, f, i, l, o, r, u, x, A, D, G, J, M, P, S$\}$, $Y = \{$1, 4, 7, a, d, g, j, m, p, s, v, y, B, E, H, K, N, Q, T$\}$, $B = \{$2, 5, 8, b, e, h, k, n, q, t, w, z, C, F, I, L, O, R, U$\}$. The number of blocks of each colour type is $RRY = 152$, $RRB = 19$, $RYY = 19$, $RBB = 152$, $YYB = 152$, $YBB = 19$, $RYB = 19$. This system is isomorphic to the cyclic STS(57) with starter blocks {0, 1, 3}, {0, 4, 9}, {0, 6, 13}, {0, 8, 26}, {0, 10, 33}, {0, 11, 32}, {0, 12, 40}, {0, 14, 41}, {0, 15, 35}, {0, 19, 38}. See, for example, Colbourn & Dinitz [10], section IV.10.1. The system has full automorphism group of order 57.

## 3.6 Type-I systems with $v \equiv 1 \pmod 6$

This section gives details (using the same compact notation as above) of type-I STS($v$)s for $v = 25$ (two systems), 31 (two systems), 37, 43, 49 and 55; STS(25) #1, STS(31) #1 and the STS(37) are used as starter systems for the proof of Theorem 3.1.1. System STS(25) #2 is required for Theorem 3.1.2.

STS(25) #1

29omijklnf ghabljhico fmnblakijn mhgomnldeo ijkdhecajk
necobkijom fgdnngkmlh fgljkmoojn imkjhlelim kmononolno

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(9, 8, 8)$ and is given by $R = \{$1, 5, a, b, d, e, g, h, k$\}$, $Y = \{$0, 8, f, i, l, m, n, o$\}$, $B = \{$2, 3, 4, 6, 7, 9, c, j$\}$. The number of blocks of each colour type is $RRY = 17$, $RRB = 19$, $RYY = 15$, $RBB = 13$, $YYB = 13$, $YBB = 15$, $RYB = 8$. This 3-colouring induces a (3, 2, 2) 3-colouring on the sub-STS(7) {0, 3, 8, 9, e, h, k}. The system has trivial full automorphism group.

STS(25) #2

`2587cedkij on876edcjk ino75ifdce jkmo6kjofi lmnolhgjkm`

`n8ngilhfom afgnmhlohm kjgonlimkn jolblmionl nemoolnmhk`

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(9,8,8)$ and is given by $R = \{$0, 1, 3, 6, a, f, g, n, o$\}$, $Y = \{$2, 4, 7, 8, c, d, i, m$\}$, $B = \{$5, 9, b, e, h, j, k, l$\}$. The number of blocks of each colour type is $RRY = 17$, $RRB = 19$, $RYY = 15$, $RBB = 13$, $YYB = 13$, $YBB = 15$, $RYB = 8$. This 3-colouring induces an equitable 3-colouring on the sub-STS(9) $\{$0, 1, 2, 9, a, b, c, d, e$\}$. The system has trivial full automorphism group.

STS(31) #1

`edcba98uts rqpon3579b dhutsrqpo6 7abejiutsr qpe6dbalkj`

`utsrqgcdmn lkutsrdcep onmlutsecd rqponmutes ctrqponugt`

`srqpouihts rqpuekjitr qumlkjtsru onmlktsuqp onmltusrqp`

`onmus`

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(11,10,10)$ and is given by $R = \{$3, 6, 9, b, c, f, g, j, p, s, u$\}$, $Y = \{$0, 1, 7, a, d, h, k, m, q, t$\}$, $B = \{$2, 4, 5, 8, e, i, l, n, o, r$\}$. The number of blocks of each colour type is $RRY = 29$, $RRB = 26$, $RYY = 21$, $RBB = 24$, $YYB = 24$, $YBB = 21$, $RYB = 10$. This 3-colouring induces a (3, 2, 2) 3-colouring on the sub-STS(7) $\{$0, 2, 4, 6, 9, b, d$\}$. The system has trivial full automorphism group.

STS(31) #2

`74cb8aopqr nlktu589cb hurinqmts6 9acsjmlipo ru7abjqlhk`

`rupt8bcmog ljutqs9cfs qportunaej otpmnsubqk ojrutpscpl`

`nsqtoruirt kmupqsgtkj mspruuhsmq tnornmhusp ortltinuts`

`urqsq`

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(11,10,10)$ and is given by $R = \{$2, 4, 6, 9, a, e, g, h, p, s, u$\}$, $Y = \{$0, 3, 5, 8, l, n, o, q, r, t$\}$, $B = \{$1, 7, b, c, d, f, i, j, k, m$\}$. The number of blocks of each colour type is $RRY = 31$, $RRB = 24$, $RYY = 19$, $RBB = 26$, $YYB = 26$, $YBB = 19$,

$RYB = 10$. This 3-colouring induces a (5, 4, 4) 3-colouring on the sub-STS(13) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c}. The system has trivial full automorphism group.

STS(37)

```
ideykflgmh noqzxwvAjb k7Amhvgpnl qzyxwkflgm hnciqsrzyx
Algmuncior jtszyAmhnc iqjurswvAz nidjeyotxw vuAidjeklp
zyxwvjekfA gxrqzywkfl AtsrzyxlgA vutszymAnx wutzdAzyxw
vupvwrxstz uvqrxsytzw whsytzoxxs ytzupwzoup vtAzupvqyu
pvqtAvowrA wrxAxsyAyA uA
```

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(13, 12, 12)$ and is given by $R =$ {d, h, p, q, s, t, u, v, w, x, y, z, A}, $Y =$ {0, 1, 2, 4, 5, 6, 8, 9, a, b, o, r}, $B =$ {3, 7, c, e, f, g, i, j, k, l, m, n}. The number of blocks of each colour type is $RRY = 44$, $RRB = 34$, $RYY = 28$, $RBB = 38$, $YYB = 38$, $YBB = 28$, $RYB = 12$. This 3-colouring induces a (3, 2, 2) 3-colouring on the sub-STS(7) {3, a, d, i, o, x, A}. The system has trivial full automorphism group.

STS(43)

```
lfmgnhwipj qkrsCFEDBA GmgnhoipjD kretxuFECB Gnhoipjqkr
eluwvFEDCG oipzqkrelf vFyxwEDGpj qkrelfmwAz yxFEGqkrel
fmgxCBAzyF GrelfmgnyE DCBAzGlfCg nhAmDtFEGB mgnhoGvuFE
DCBnhoiGxw vFEDCoipAF GzyxwEpjGB AzyxFEqGDC BAzyFGFEDC
BAzztGuBvC wDxEyFAuBv CwDxEyFsBv CwDxEyFszC wDxEyFsztD
xEyFsztAEy sztAuFsztA uBztAuBvGA uBvCGBvCwG rwGDGCEDGD
F
```

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(15, 14, 14)$ and is given by $R =$ {1, 5, e, f, g, i, j, k, l, m, n, p, q, r, G}, $Y =$ {0, 2, 3, 4, 6, 7, 8, 9, a, b, c, d, D, F}, $B =$ {h, o, s, t, u, v, w, x, y, z, A, B, C, E}. The number of blocks of each colour type is $RRY = 38$, $RRB = 67$, $RYY = 60$, $RBB = 31$, $YYB = 31$, $YBB = 60$, $RYB = 14$. The system has trivial full automorphism group.

STS(49)

```
Ehpiqjrksl tmunvwMLKJ IHGFpiqjrk sltmunvgxG yLKJIHMqjr
kslAmunvgo yGMzLKJIHr ksltmunvgo hzCBALKJIM sltmunvgoh
pAEDCBLKJM tmunvgohpi BGFEDCLKMu nJgxhpiqCL HIGFEDMvgo
hpiqjDKJIH GFEMChpiqj rMHxLKJIGF piqjrkMtLz yKIHGqjrks
MBAzLKJIHr kslMCDBALK JIsltMFEDC BLKJtmMHGF EDCLKuMJIH
GFEDLMLKJI HGFEExFyGz HvIBJCKDLF yGzHAIBJCK DLwGzHAIBJ
CKDLwEHAIB JCKDLwExIB oKDLwExFJJ CKDLwExFyK DLwExFyGwE
xFyGzLExFy zAFyGzHAMG zHAIMHAIBM IBJMJCMKMM HM
```

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(17, 16, 16)$ and is given by $R = \{$0, 1, 2, 3, 4, 6, 7, 8, 9, a, b, c, d, e, f, H, L$\}$, $Y = \{$5, h, i, j, k, l, m, o, p, q, r, s, t, u, v, M$\}$, $B = \{$g, n, w, x, y, z, A, B, C, D, E, F, G, I, J, K$\}$. The number of blocks of each colour type is $RRY = 95$, $RRB = 41$, $RYY = 33$, $RBB = 87$, $YYB = 87$, $YBB = 33$, $RYB = 16$. The system has trivial full automorphism group.

STS(55)

```
rjsktlumvn eoxyqOAzER QPNMLKSskt luzvnwoxpy qmiBCRQPON
MLStlumvnw oxpyqzirCE DRQPONMSum vnwoxpyqzi rjDGFERQPO
NSvnwoxpyq zirjsEIHGF RQPOSwoxpy qzirjskFKJ IHGRQPSxpy
qzirjKktGC MLJIHRQSyq HirjsktlpO NMLKJIRSzi rjsktluIQP
ONMLKJSrjs ktlumSBRQP ONMLKsktlu zvSODCRQPN MLtlmvnSCD
FERQPONMum vnwSHGFERQ PONvnwoSJI HGFRQPOwox SHELKJIGRQ
PxpSDNMLJI HRQySPONML KJIRSDRQPN MLKJJBKCLD MENFOGPHQI
RKCLDMENFO GPHQIRALDM ENFOGPHQIR AJMENFOGPH QIRAJBNFGP
HQIRAJBKOG PHQIRAJBKC PHQIRAJBKC LQIRAJBKCL ORAJBKCLDM
JBKCLDMESK wLMENSLDME NFSMNFOSNF OGSOGPSPHS QSSKO
```

The 3-colouring is unique, modulo swapping colours $Y$ and $B$, with colour class sizes $(19, 18, 18)$ and is given by $R = \{$1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f, g, h, D, O$\}$, $Y = \{$0, i, j, k, l, m, n, o, p, q, r, s, t, v, w, x, y, S$\}$, $B = \{$u, z, A, B, C, E, F, G, H, I, J, K, L, M, N, P, Q, R$\}$. The number of blocks of each colour type is $RRY = 126$,

$RRB = 45$, $RYY = 36$, $RBB = 117$, $YYB = 117$, $YBB = 36$, $RYB = 18$. The system has trivial full automorphism group.

## 3.7 Type-II systems with $v \equiv 3 \pmod 6$

Type-II STS($v$)s for $v = 27$ (two systems), 33 (three systems), 39, 45 and 51.

STS(27) #1

`2687dhiqpl mko876gpcn hmkqo768io flqkpjn5bd egqpomnajd mhnloqonqh lgjmp8ckhp nolqmplgke qmonfjlpmo kiqjnqmefi qonpjnokhj iqpqlpp`

The 3-colouring is unique with colour class sizes $(10, 9, 8)$ and is given by $R = \{$0, 3, 7, 8, b, c, d, f, h, n$\}$, $Y = \{$2, 4, 5, a, g, j, k, l, q$\}$, $B = \{$1, 6, 9, e, i, m, o, p$\}$. The number of blocks of each colour type is $RRY = 24$, $RRB = 21$, $RYY = 17$, $RBB = 15$, $YYB = 19$, $YBB = 13$, $RYB = 8$. This 3-colouring induces a (4, 3, 2) 3-colouring on the sub-STS(9) $\{$0, 1, 2, 3, 4, 5, 6, 7, 8$\}$. The system has trivial full automorphism group.

STS(27) #2

`74cb8apiol mkq589cbok nhmpq69acl pjniqo7abq fojkpn8bcm qiponl9ckj miqnpahmqj lpobiopqnl mcnglkoqpe kmljpqjhgp noqglhnmoq fnkpqom`

The 3-colouring is unique with colour class sizes $(10, 9, 8)$ and is given by $R = \{$2, 4, 5, 9, b, e, h, k, m, n$\}$, $Y = \{$6, 7, 8, c, d, g, l, o, p$\}$, $B = \{$0, 1, 3, a, f, i, j, q$\}$. The number of blocks of each colour type is $RRY = 24$, $RRB = 21$, $RYY = 17$, $RBB = 15$, $YYB = 19$, $YBB = 13$, $RYB = 8$. The 3-colouring induces a (5, 4, 4) 3-colouring on the sub-STS(13), $\{$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c$\}$. This system is used with Lemma 3.2.7. The system has trivial full automorphism group.

STS(33) #1

`ed8bc9ahps krvtqw3579 bdwqjnpvos u689cejkup swvtra7bde`
`mtkvpqowuc devniurtsw qeadgswqmu prvcdlsojn qrwvbesmlo`
`wquvteniml tvswucouwq mlstvqrnls utpwpovurw mqtklrtonw`
`uvthknpruv wuwpmtvors rvtswu`

The 3-colouring is unique with colour class sizes $(12, 11, 10)$ and is given by $R$ = {0, 2, 8, 9, c, e, f, g, m, r, s, t}, $Y$ = {4, 5, 7, a, b, j, k, l, n, o, v}, $B$ = {1, 3, 6, d, h, i, p, q, u, w}. The number of blocks of each colour type is $RRY = 35$, $RRB = 31$, $RYY = 26$, $RBB = 24$, $YYB = 29$, $YBB = 21$, $RYB = 10$. The 3-colouring induces a (6, 5, 4) 3-colouring on the sub-STS(15) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e}. The system has trivial full automorphism group.

STS(33) #2

`h3uvpoqjik swtmlruvgw oqpikjrtsl n5cpokjisw nmltvwusrt`
`mlnjikpoqr tslnmikjop vtsrnmlkji qpo9wuvlnj ikquvwlnmi`
`kjvbnmlkji wupoqsrtoq prtsqpotsr fuwmvrvuvh uwqvrvtsow`
`uouvpwtnwq tuovswtuwu urvwsw`

The 3-colouring is unique with colour class sizes $(12, 11, 10)$ and is given by $R$ = {1, 3, 7, a, d, f, j, m, o, s, u, w}, $Y$ = {2, 4, 8, b, e, g, k, n, p, t, v}, $B$ = {0, 5, 6, 9, c, h, i, l, q, r}. The number of blocks of each colour type is $RRY = 30$, $RRB = 36$, $RYY = 31$, $RBB = 19$, $YYB = 24$, $YBB = 26$, $RYB = 10$. The 3-colouring induces a (4, 3, 2) 3-colouring on the sub-STS(9) {6, 7, 8, 9, a, b, u, v, w}. The system has trivial full automorphism group.

STS(33) #3

`59v6uwklmn opqrst84vw ulmfoprstk qw7vumropq nstklu89no`
`pqrstklmwu vopqrstklm n79pqrstkl mnowqkstlr mnopvrstkm`
`gopqnstkln mopqrtnlmk opqrsjhwiv uvhuwijwnu guvjviqwvu`
`wovwusturv wowrttvusw uvvuww`

The 3-colouring is unique with colour class sizes $(12, 11, 10)$ and is given by $R$ = {3, 4, 5, 6, b, h, i, j, r, s, t, v}, $Y$ = {7, 8, 9, a, c, d, n, o, p, q, u}, $B$ = {0, 1, 2, e, f, g, k, l, m, w}. The number of blocks of each colour type is $RRY = 30$, $RRB = 36$, $RYY = 31$, $RBB = 19$, $YYB = 24$, $YBB = 26$, $RYB = 10$. The 3-colouring induces a (5, 4, 4) 3-colouring on the sub-STS(13) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, u, v, w}. This system is used with Lemma 3.2.7. The system has trivial full automorphism group.

STS(39)

```
a7Ct9bAsrB vuwyxzpoq4 67BACrtsuw vxzyoqpAC9 aBtsrwvuzy
xqpo98Bbpo qyxzvuwsrt aAbCoqpxzy uwvrBsBAqp ozyxwvutsr
8Cyxzsrtpo qvuwCAxzyr tsoqpuwvbz yxtsrqpowv uBvuwpoqsr
tyxzuwvoqp rtsxzywvuq potsrzyxmj CBlnAgijBA CAClntlkBn
mAnCBAkCCA mByvCBxzAs uvBACACxyB xwBzyAzCBA wCCAzBC
```

The 3-colouring is unique with colour class sizes $(14, 13, 12)$ and is given by $R$ = {0, 3, 6, 9, f, i, k, l, o, r, u, x, A, B}, $Y$ = {1, 4, 7, a, c, g, j, m, p, s, v, y, C}, $B$ = {2, 5, 8, b, d, e, h, n, q, t, w, z}. The number of blocks of each colour type is $RRY = 58$, $RRB = 33$, $RYY = 27$, $RBB = 45$, $YYB = 51$, $YBB = 21$, $RYB = 12$. This 3-colouring induces a $(6, 5, 4)$ 3-colouring on the sub-STS(15) {o, p, q, r, s, t, u, v, w, x, y, z, A, B, C}. The system has trivial full automorphism group.

STS(45)

```
ace89bdwuv FDEIGHCABz xy96n7ediu vwEFGHABCx yzI5d8bevw
utFDHIGBCA yzEscbzdCA BDxywuvIGH FEbaeEwABC xyzuvGHIDF
edcBCAyzxv wuHIGEFD9d FDEIGHCABz xywuvaDEFG HIABCxyzuv
wcgDHIGBCA yzxvwuFbet xyCABFnwuv IGHxyzABCD EFuvwGHIey
zxBCAEFDvw uHIGIGHuvz xyFDECABGH IuvwxyzpFA BCEHIGvwuy
zxEqBCAFrz noqpEolrmt Fsksnqtorq sEqpttsros DrEIDttHEG
ICDFHDAGBI HzHCFDGFFE IIHGDHEGII
```

The 3-colouring is unique with colour class sizes $(16, 15, 14)$ and is given by $R$ = {2, 5, 8, 9, b, e, g, j, m, p, s, v, y, B, E, H}, $Y$ = {0, 3, 6, c, f, h, k, n, q, t, u, x, A, D, G}, $B$ = {1, 4, 7, a, d, i, l, o, r, w, z, C, F, I}. The number of blocks of each colour type is $RRY = 35$, $RRB = 85$, $RYY = 78$, $RBB = 20$, $YYB = 27$,

$YBB = 71$, $RYB = 14$. The system has trivial full automorphism group.

STS(51)

```
firBsktlum vnwopyMxON LKJIHrjskt lumvnwoxhz DMAONLKJIs
ktlumvnwox phHAKCBONM LJtlumvnwo xphqiBEDCO NMLKumvnwo
xphqirDCGF EONMLvnwox phqirjDIHG FEONMwoxph qirjsEKJIH
GFONxphqir jskFMLKJIH GOhqirjskt GONMLKJIHi rjsktlHzON
MLKJIjsktl uIBAONMLKJ ktlumJDCBO NMLKlumvKF EDCONMLmvn
LHGFEDONMn wMJIHGFEON oHNCLKJIGF OEONMLKJIG HzIAJBKCLD
MENFOGIAJB KCLDMENFOG yJBKGLrCNF OEyHKCLDMw NFOGyzHLDM
ENFOGyHzIM ENFOGyHzIA NFOCyHzIAJ OGyHzIAJBy zIAJBKzIAJ
BAJBGELKBK CLDCLDMDME KNFHM
```

The 3-colouring is unique with colour class sizes $(18, 17, 16)$ and is given by $R = \{$0, 3, f, h, i, j, k, l, m, n, o, p, s, t, u, v, w, x$\}$, $Y = \{$1, 2, 4, 5, 6, 7, 8, 9, a, b, c, d, e, g, z, D, K$\}$, $B = \{$q, r, y, A, B, C, E, F, G, H, I, J, L, M, N, O$\}$. The number of blocks of each colour type is $RRY = 58$, $RRB = 95$, $RYY = 87$, $RBB = 41$, $YYB = 49$, $YBB = 79$, $RYB = 16$. The system has trivial full automorphism group.

## 3.8 Type-II systems with $v \equiv 1 \pmod 6$

Type-II STS($v$)s for $v = 25$, 31 (two systems), 37, 43 and 49. The STS(25), the first STS(31) and the STS(37) each have a correctly aligned sub-STS(7) and therefore they can be used as starter systems in the proof of Theorem 3.1.2.

STS(25)

```
149mijkbfo hn9abgjhin kfobakilhn gomnocdeij kmldecomjk
iecdbkjnnm fglooglfmh fgnnkjmljh iooijmnelm lmlohnonko
```

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(9, 9, 7)$ and is given by $R = \{$0, 1, 2, a, h, i, j, n, o$\}$, $Y = \{$3, 4, 5, b, c, e, f, g, m$\}$, $B = \{$6, 7, 8, 9, d, k, l$\}$. The number of blocks of each colour type is $RRY = 18$, $RRB = 18$, $RYY = 19$, $RBB = 10$, $YYB = 17$, $YBB = 11$, $RYB = 7$. This 3-colouring induces a (3, 3, 1) 3-colouring on the sub-STS(7) $\{$1, 3, 8, a, e, f, i}. The system has trivial full automorphism group.

STS(31) #1

`edcba9gkus rqont3579b dhutsrqpo6 7abeliukrq ptn6tmafeq jupsricdmf klutsrdcep onmlutsedk mrtponusqe cdsrponuog tsrpuihtsr qpuskjitrq uhljtsrujn ltsquqponm lturqponmu otsmq`

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(11, 11, 9)$ and is given by $R = \{$1, 3, 7, 8, a, d, f, j, l, o, u$\}$, $Y = \{$4, 6, 9, c, e, h, i, n, r, s, t$\}$, $B = \{$0, 2, 5, b, g, k, m, p, q$\}$. The number of blocks of each colour type is $RRY = 27$, $RRB = 28$, $RYY = 29$, $RBB = 17$, $YYB = 26$, $YBB = 19$, $RYB = 9$. This 3-colouring induces a (3, 3, 1) 3-colouring on the sub-STS(7) {1, 4, 5, 8, 9, c, d}. The system has trivial full automorphism group.

STS(31) #2

`74cb8akmun iltrs589cb ronmuqkpt6 9acijthlus rq7absgkpn rutq8bcjpi rmotqu9cgs hotpqnuauh okjsprtbnl ptoksrucqk ljrtspuhqj pmsroumigl outsrorqut spmnltmqsr unpftuslsn quqto`

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(11, 11, 9)$ and is given by $R = \{$0, 2, 5, 7, 8, f, j, k, n, q, r$\}$, $Y = \{$1, 3, 4, 6, a, d, g, h, i, o, t$\}$, $B = \{$9, b, c, e, l, m, p, s, u$\}$. The number of blocks of each colour type is $RRY = 27$, $RRB = 28$, $RYY = 29$, $RBB = 17$, $YYB = 26$, $YBB = 19$, $RYB = 9$. The 3-colouring induces a (5, 5, 3) 3-colouring on the sub-STS(13) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c}. The system has trivial full automorphism group.

STS(37)

`3Akgsrtvuw mlijzqyx4A rtsumvlnwi kjxzy5tsrw vunmlkjizy xAgfhsrtpo qvuwyxzcfh grtiqpwxzv uyhiftsrpo vuzyxw9Aby xzpoqvuwaA xzvoqpuwyb zyxqpowvuA jinyxkpozq kjxzywopkj szyxqpofAh ovuwgAuwvh yruAsqtvuw urtswvtsrw vulAnyxzmA xzynzvxArA tsAtAxAzyA zA`

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(13, 13, 11)$ and is given by $R = \{$0, 4, 6, 9, b, c, d, g, n, p, t, u, z$\}$, $Y = \{$2, 3, 8, e, f, i, j, m, o, s, w, y, A$\}$, $B = \{$1, 5, 7, a, h, k, l, q, r, v, x$\}$. The number of blocks of each colour type is $RRY = 47$, $RRB = 31$, $RYY = 32$, $RBB = 35$, $YYB = 46$, $YBB = 20$, $RYB = 11$. This 3-colouring induces a (3, 3, 1) 3-colouring on the sub-STS(7) $\{$c, d, e, f, g, h, A$\}$. The system has trivial full automorphism group.

STS(43)

```
lnmgfhoipj qkrsFEDCBA Gmgnhoipjq kretuFEDCB Gvhoipjqkf
elutwFEDCG oipjqkrelf wuyxFEDGpj qkrelfmwAz yxFEGqknel
AmgxuCBzyF GrelfmgnyE DCBAzGlfmy nhGBtFEDCA mgrhoGvuFE
DCBdioGnxw vFEDCoiGpz yxwFEDpjhG BvzxFEqGDC BAzyFCGvFE
DBAzztiBvC wDxEyFAAuB vCwDxEyFsv CwDxEyFszr DxEFszACBD
xEyFsztAGE yFsztAwFsz tAuBztAuBv GAuBvCGBCw yCwDGDxGEG
G
```

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(15, 15, 13)$ and is given by $R = \{$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, u, D$\}$, $Y = \{$i, j, s, t, v, w, x, y, z, A, B, C, E, F, G$\}$, $B = \{$d, e, f, g, h, k, l, m, n, o, p, q, r$\}$. The number of blocks of each colour type is $RRY = 37$, $RRB = 68$, $RYY = 69$, $RBB = 23$, $YYB = 36$, $YBB = 55$, $RYB = 13$. The system has trivial full automorphism group.

STS(49)

```
9648bMpoqv uwEDFBACHz IKJL4Mba8o ypuwvDFEAC BxGIHKL5Ma
bqpowvuFED CBAIHGLKJ7 bFzHGIpoqK JLvuwBACED 69MbGIHoqp
JLKuwvACBD FEaMIHGqpo LKJwvuCBAF ED9MsrtEDF yxzHGIBAwK
JLrtsDFExz yGIHACBJLK tsrFEDzyxI HGCBALKJmK qLBACsrtyz
EDHGIJJLKA CBrtsxyDFE GIHLKJCBAt srzyxFEDIH GligknMEDF
BACgMnmkDF EABJhMmnFE DCBAjnmKJL HGIilMnJLK GIHmMLKJIH
GlMHGIBACG IHACBIHGCB AKJLEDFJLK DFEzLKJFED xuswzMsMzy
wtMCFvzyux yMyMxMMLJG EILMEMLKIF MKHLKGJMLK MJ
```

The 3-colouring is unique, modulo swapping colours $R$ and $Y$, with colour class sizes $(17, 17, 15)$ and is given by $R = \{0, 3, 6, 9, c, f, i, l, o, q, t, z, A, D, G, J, M\}$, $Y = \{1, 4, 7, a, d, g, j, m, p, r, s, u, x, B, E, H, K\}$, $B = \{2, 5, 8, b, e, h, k, n, v, w, y, C, F, I, L\}$. The number of blocks of each colour type is $RRY = 73$, $RRB = 63$, $RYY = 64$, $RBB = 57$, $YYB = 72$, $YBB = 48$, $RYB = 15$. The system has trivial full automorphism group.

Table 3.1: Constructions for type-I STS$(6k+3)$

| {3}-GDD type | STS($v$)s | Residue (mod 72) | Exceptions |
|---|---|---|---|
| $8^{3t}$ | 27 | 3 | - |
| $8^{3t+1}18^1$ | 27, 57 | 9 | 81 |
| $8^{3t}28^1$ | 27, 87 | 15 | 87, 159 |
| $8^{3t+2}14^1$ | 27, 45 | 21 | 93 |
| $8^{3t+1}$ | 27 | 27 | 27 |
| $8^{3t}10^1$ | 27, 33 | 33 | 33 |
| $8^{3t}12^1$ | 27, 39 | 39 | 39 |
| $8^{3t}14^1$ | 27, 45 | 45 | 45 |
| $8^{3t}16^1$ | 27, 51 | 51 | 51 |
| $8^{3t}18^1$ | 27, 57 | 57 | 57, 129 |
| $8^{3t+1}12^1$ | 27, 39 | 63 | 63 |
| $8^{3t}22^1$ | 27, 69 | 69 | 69, 141 |

Table 3.2: Constructions for type-I STS$(6k+1)$

| {3}-GDD type | STS($v$)s | Residue (mod 54) | Exceptions |
|---|---|---|---|
| $6^{3t+1}10^1$ | 25, 37 | 1 | 55 |
| $6^{3t}$ | 25 | 7 | - |
| $6^{3t+2}8^1$ | 25, 31 | 13 | 67 |
| $6^{3t+2}10^1$ | 25, 37 | 19 | 73 |
| $6^{3t+1}$ | 25 | 25 | 25 |
| $6^{3t}8^1$ | 25, 31 | 31 | 31 |
| $6^{3t}10^1$ | 25, 37 | 37 | 37 |
| $6^{3t+2}$ | 25 | 43 | 43 |
| $6^{3t+1}8^1$ | 25, 31 | 49 | 49 |

Table 3.3: Constructions for type-I exceptions

| | {3}-GDD type | STS($v$)s | $u$ | $u_1 u_2 u_3$ |
|---|---|---|---|---|
| 63 | $6^3$ | 27 | 9 | 3 3 3 |
| 69 | $6^3$ | 33 | 15 | 5 5 5 |
| 81 | $6^4$ | 27 | 9 | 3 3 3 |
| 87 | $8^3$ | 39 | 15 | 5 5 5 |
| 93 | $10^3$ | 33 | 3 | 1 1 1 |
| 129 | $10^3 12^1$ | 33, 39 | 3 | 1 1 1 |
| 141 | $12^3 10^1$ | 39, 33 | 3 | 1 1 1 |
| 159 | $14^3 10^1$ | 45, 33 | 3 | 1 1 1 |
| 67 | $6^3$ | 31 #2 | 13 | 5 4 4 |
| 73 | $8^3$ | 25 | 1 | 1 0 0 |

Table 3.4: Constructions for type-II STS($6k+3$)

| {3}-GDD type | STS($v$)s | Residue (mod 72) | Exceptions |
|---|---|---|---|
| $8^{3t}$ | 27 | 3 | - |
| $8^{3t+1} 42^1$ | 27, 129 | 9 | 81, 153, 225 |
| $8^{3t} 52^1$ | 27, 159 | 15 | 87, 159, 231, 303 |
| $8^{3t+2} 14^1$ | 27, 45 | 21 | 93 |
| $8^{3t+1}$ | 27 | 27 | 27 |
| $8^{3t} 10^1$ | 27, 33 | 33 | 33 |
| $8^{3t} 12^1$ | 27, 39 | 39 | 39 |
| $8^{3t} 14^1$ | 27, 45 | 45 | 45 |
| $8^{3t} 16^1$ | 27, 51 | 51 | 51 |
| $8^{3t} 42^1$ | 27, 129 | 57 | 57, 129, 201, 273 |
| $8^{3t+1} 12^1$ | 27, 39 | 63 | 63 |
| $8^{3t} 46^1$ | 27, 141 | 69 | 69, 141, 213, 285 |

Table 3.5: Constructions for type-II STS($6k+1$)

| {3}-GDD type | STS($v$)s | Residue (mod 54) | Exceptions |
|---|---|---|---|
| $6^{3t+1} 10^1$ | 25, 37 | 1 | 55 |
| $6^{3t}$ | 25 | 7 | - |
| $6^{3t+2} 8^1$ | 25, 31 | 13 | 67 |
| $6^{3t+2} 10^1$ | 25, 37 | 19 | 73 |
| $6^{3t+1}$ | 25 | 25 | 25 |
| $6^{3t} 8^1$ | 25, 31 | 31 | 31 |
| $6^{3t} 10^1$ | 25, 37 | 37 | 37 |
| $6^{3t+2}$ | 25 | 43 | 43 |
| $6^{3t+1} 8^1$ | 25, 31 | 49 | 49 |

Table 3.6: Constructions for type-II exceptions

| | {3}-GDD type | STS($v$)s | $u$ | $u_1u_2u_3$ |
|---|---|---|---|---|
| 63 | $6^3$ | 27 #1 | 9 | 4 3 2 |
| 69 | $6^3$ | 33 #1 | 15 | 6 5 4 |
| 81 | $8^3$ | 33 #2 | 9 | 4 3 2 |
| 87 | $8^3$ | 39 | 15 | 6 5 4 |
| 93 | $10^3$ | 33 | 3 | 2 1 0 |
| 129 | $10^3 12^1$ | 33, 39 | 3 | 2 1 0 |
| 141 | $12^3 10^1$ | 39, 33 | 3 | 2 1 0 |
| 153 | $14^3 8^1$ | 45, 27 | 3 | 2 1 0 |
| 159 | $14^3 10^1$ | 45, 33 | 3 | 2 1 0 |
| 201 | $10^5 16^1$ | 33, 51 | 3 | 2 1 0 |
| 213 | $10^7$ | 33 | 3 | 2 1 0 |
| 225 | $10^6 14^1$ | 33, 45 | 3 | 2 1 0 |
| 231 | $10^6 16^1$ | 33, 51 | 3 | 2 1 0 |
| 273 | $10^9$ | 33 | 3 | 2 1 0 |
| 285 | $10^7 24^1$ | 33, 75 | 3 | 2 1 0 |
| 303 | $10^{10}$ | 33 | 3 | 2 1 0 |
| 67 | $6^3$ | 31 #2 | 13 | 5 5 3 |

# Chapter 4

# Independent sets in Steiner triple systems

## 4.1 Introduction

Recall that a subset $U$ of $V$ in an STS($v$), $S = (V, \mathcal{B})$, is an *independent set* if no three points of $U$ occur in a single block $T \in \mathcal{B}$. We denote by $I_k(S)$ the number of independent sets of cardinality $k$ that occur in $S$. If there is no risk of confusion, we write $I_k$ or $I_k(v)$ instead of $I_k(S)$.

The main purpose of this chapter is to obtain a formula for $I_k(S)$ in terms of the numbers of occurrences in $S$ of certain configurations. This is stated as Theorem 4.3.1. From this formula we obtain explicit expressions for $I_k(v)$, $3 \leq k \leq 8$.

Sauer and Schönheim [65] show that an STS(21) cannot have an independent set of cardinality greater than ten. Moreover, with a suitable implementation of Stinson's hill-climbing algorithm [68], it is not too difficult to construct for each $k \in \{8, 9, 10\}$ an STS(21) having an independent set of maximum cardinality $k$ (Colbourn and Rosa [16], section 17.2). The only remaining possibility is that there might exist an STS(21) whose largest independent set has fewer than eight points. However, once we have obtained the formula for $I_8(v)$ we can prove that every Steiner triple system of order 21 has an independent set of cardinality eight (Theorem 4.4.1).

Theorem 4.4.1 then allows us to solve the problem of determining those values of $\chi$ for which there exists an STS(21) with chromatic number $\chi$. It is well known that STS(21)s with chromatic number 3 exist. Haddad [39] constructs an STS(21) with chromatic number 4, and Forbes, Grannell and Griggs [26] prove that every STS(21) is 5-colourable. Using Theorem 4.4.1 it is relatively straightforward to improve this last result and thereby completely determine the spectrum of chromatic numbers for STS(21)s. We shall prove that every Steiner triple system of order 21 is 4-colourable (Theorem 4.4.2).

The chapter concludes with two results concerning the occurrence of maximum independent sets in a Steiner triple system of order $v$ in the extreme cases, where the sets have size $(v+1)/2$ for $v \equiv 3$ or $7 \pmod{12}$, or $(v-1)/2$ for $v \equiv 1$ or $9 \pmod{12}$ (Theorems 4.5.1 and 4.5.2).

## 4.2 Configurations

Recall that a configuration, $\mathcal{X}$ is a collection of 3-element sets, called *blocks*, such that a pair of distinct elements appears in at most one block. We denote the number of points in $\mathcal{X}$ by $p(\mathcal{X})$ and the number of blocks by $b(\mathcal{X})$. If $P$ is a point of $\mathcal{X}$, we call the number of blocks of $\mathcal{X}$ containing $P$ the *degree* of $P$.

If $S$ is a Steiner triple system and $\mathcal{X}$ is a configuration, we denote by $n(\mathcal{X}, S)$ the number of occurrences of $\mathcal{X}$ in $S$. If the system $S$ is fixed, we usually abbreviate $n(\mathcal{X}, S)$ to $n(\mathcal{X})$. If $\mathcal{X}$ is denoted by a subscripted upper-case letter, $X_i$, say, we usually write the corresponding subscripted lower-case letter, $x_i$, for $n(X_i)$. A configuration $X$ whose number of occurrences in an STS($v$) depends only upon $v$ and not on the actual STS($v$) is *constant*, otherwise it is *variable*.

We will need details of the 31 configurations of at most eight points, and for convenience we list them in Table 4.1. For brevity, set brackets and commas have been omitted. The numbering assigned to the configurations is standard [16, 17].

Of particular relevance are configurations in the table that have no points of degree 1. There are precisely nine of them:

$$C_{16}, D_1, E_1, E_2, E_3, F_1, F_2, F_3, G_1, \tag{4.1}$$

and they are all variable. The main reason for our interest in these configurations is a theorem established by Horák, Phillips, Wallis and Yucas [42]: *Any constant n-block configuration, together with all m-block configurations for $m \leq n$ having all points of degree at least two form a generating set for the n-block configurations.* In fact, the expression for the frequency of the $n$-block configuration has the form

$$n(X,S) = Q_0(v) + \sum_{\mathcal{G}} Q_{\mathcal{G}}(v)\, n(\mathcal{G},S), \tag{4.2}$$

where the sum is over all configurations $\mathcal{G}$ that have no more than $n$ blocks and have no points of degree 1, and $Q_0, Q_{\mathcal{G}} \in \mathbb{Q}[v]$. We will have more to say about formula (4.2) in Chapter 7.

Table 4.1: Configurations of up to eight points

| | | |
|---|---|---|
| $A_0$ | 012 | $\lvert\text{Aut}\rvert = 6$ |
| $A_1$ | 012 345 | $\lvert\text{Aut}\rvert = 72$ |
| $A_2$ | 012 034 | $\lvert\text{Aut}\rvert = 8$ |
| $B_2$ | 012 034 567 | $\lvert\text{Aut}\rvert = 48$ |
| $B_3$ | 012 034 056 | $\lvert\text{Aut}\rvert = 48$ |
| $B_4$ | 012 034 156 | $\lvert\text{Aut}\rvert = 8$ |
| $B_5$ | 012 034 135 | $\lvert\text{Aut}\rvert = 6$ |
| $C_{10}$ | 012 034 156 357 | $\lvert\text{Aut}\rvert = 8$ |
| $C_{11}$ | 012 034 135 067 | $\lvert\text{Aut}\rvert = 4$ |
| $C_{12}$ | 012 034 135 267 | $\lvert\text{Aut}\rvert = 4$ |
| $C_{14}$ | 012 034 135 246 | $\lvert\text{Aut}\rvert = 4$ |
| $C_{15}$ | 012 034 135 236 | $\lvert\text{Aut}\rvert = 6$ |
| $C_{16}$ (Pasch) | 012 034 135 245 | $\lvert\text{Aut}\rvert = 24$ |
| $D_1$ (mitre) | 012 034 135 236 456 | $\lvert\text{Aut}\rvert = 12$ |
| $D_2$ | 012 034 135 236 146 | $\lvert\text{Aut}\rvert = 8$ |
| $D_3$ | 012 034 135 236 147 | $\lvert\text{Aut}\rvert = 2$ |
| $D_4$ | 012 034 135 236 457 | $\lvert\text{Aut}\rvert = 2$ |
| $D_5$ | 012 034 135 245 067 | $\lvert\text{Aut}\rvert = 8$ |
| $D_6$ | 012 034 135 246 257 | $\lvert\text{Aut}\rvert = 2$ |
| $D_7$ | 012 034 135 246 567 | $\lvert\text{Aut}\rvert = 4$ |
| $E_1$ (semihead) | 012 034 135 236 146 245 | $\lvert\text{Aut}\rvert = 24$ |
| $E_2$ | 012 034 135 236 147 567 | $\lvert\text{Aut}\rvert = 2$ |
| $E_3$ (6-cycle) | 012 034 135 246 257 367 | $\lvert\text{Aut}\rvert = 12$ |
| $E_6$ | 012 034 135 236 146 057 | $\lvert\text{Aut}\rvert = 2$ |
| $E_7$ | 012 034 135 236 146 247 | $\lvert\text{Aut}\rvert = 8$ |
| $E_8$ | 012 034 135 236 147 257 | $\lvert\text{Aut}\rvert = 2$ |
| $F_1$ (STS(7)) | 012 034 135 236 146 245 056 | $\lvert\text{Aut}\rvert = 168$ |
| $F_2$ | 012 034 135 236 146 247 057 | $\lvert\text{Aut}\rvert = 4$ |
| $F_3$ | 012 034 135 236 147 257 456 | $\lvert\text{Aut}\rvert = 6$ |
| $F_{20}$ | 012 034 135 236 146 245 057 | $\lvert\text{Aut}\rvert = 8$ |
| $G_1$ | 012 034 135 236 147 257 456 067 | $\lvert\text{Aut}\rvert = 48$ |

The theorem guarantees that for each configuration in Table 4.1, there is a formula giving its frequency of occurrence in a given STS($v$) as a function of $v$

and the frequencies of the configurations (4.1). We adopt the convention of using appropriate subscripted lower-case letters for configuration frequencies, except that we write $p$ for $c_{16}$ (Pasch, $C_{16}$) and $m$ for $d_1$ (mitre, $D_1$). For brevity we write $n_v$ for $v(v-1)(v-3)$.

Formulae for the first three configurations are well-known. Indeed, the first is just the formula for the number of blocks in an STS($v$):

$$a_0 = \frac{1}{6}v(v-1), \quad a_1 = \frac{n_v}{72}(v-7), \quad a_2 = \frac{n_v}{8}.$$

The next nine equalities are taken from Grannell, Griggs and Mendelsohn [34]:

$$b_2 = \frac{n_v}{48}(v-7)(v-9), \quad b_3 = \frac{n_v}{48}(v-5), \quad b_4 = \frac{n_v}{8}(v-7), \quad b_5 = \frac{n_v}{6},$$

$$c_{10} = \frac{n_v}{8}(v-8)+3p, \quad c_{11} = \frac{n_v}{4}(v-7),$$

$$c_{12} = \frac{n_v}{4}(v-9)+12p, \quad c_{14} = \frac{n_v}{4} - 6p, \quad c_{15} = \frac{n_v}{6}.$$

The formulae for the 5-block configurations are given by Danziger, Mendelsohn, Grannell and Griggs [17]:

$$d_2 = 3p, \quad d_3 = \frac{n_v}{2} - 12p, \quad d_4 = \frac{n_v}{2} - 12p - 6m,$$

$$d_5 = 3(v-7)p, \quad d_6 = \frac{n_v}{2} - 12p, \quad d_7 = \frac{n_v}{4} - 6p - 3m.$$

Formulae are now derived for $e_6$, $e_7$, $e_8$ and $f_{20}$ using the technique described in section 13.1 of Colbourn and Rosa [16]. But note that they can also be obtained by using Theorem 7.3.1 of Chapter 7. Indeed, the formulae for $e_6$, $e_7$, $e_8$ are also listed in Appendix B.

By considering the different ways of adding a block to configuration $D_2$ linking two points of degree 2 we obtain the formula

$$e_7 = d_2 - 3e_1,$$

and by linking the point of degree 1 to a point of degree 2,

$$e_6 = 4d_2 - 12e_1.$$

Similarly, by adding a block to the mitre configuration linking two of its points of degree 2,

$$e_8 = 6m,$$

and, finally, by adding a block to $E_1$ linking two of its points,

$$f_{20} = 3e_1 - 21f_1.$$

## 4.3 Independent sets

We now state and prove the main result.

**Theorem 4.3.1** *Let $S = (V, \mathcal{B})$ be a given Steiner triple system of order $v$. Then*

$$I_k(S) = \binom{v}{k} + \sum_{\mathcal{X}} (-1)^{b(\mathcal{X})} n(\mathcal{X}, S) \binom{v - p(\mathcal{X})}{k - p(\mathcal{X})},$$

*where the sum extends over all configurations $\mathcal{X}$ consisting of at most $k$ points.*

**Proof.** If $W$ is a subset of $V$ and $\mathcal{X}$ is a configuration, denote by $n(\mathcal{X}, W)$ the number of occurrences of $\mathcal{X}$ in $W$.

Consider a $k$-element set $W$. Suppose $W$ contains exactly $l$ blocks of $S$ and we compute the sum over all possible configurations, $\mathcal{X}$, $\sum_{\mathcal{X}} (-1)^{b(\mathcal{X})} n(\mathcal{X}, W)$. Then we obtain the value $\sum_{i=1}^{l} (-1)^i \binom{l}{i} = -1$ if $l \geq 1$, and zero if $l = 0$. Hence

$$\begin{aligned} I_k(S) &= \binom{v}{k} + \sum_W \sum_{\mathcal{X}} (-1)^{b(\mathcal{X})} n(\mathcal{X}, W) \\ &= \binom{v}{k} + \sum_{\mathcal{X}} (-1)^{b(X)} \sum_W n(\mathcal{X}, W), \end{aligned}$$

where $\sum_W$ indicates a sum over all possible $k$-subsets of $V$.

But $\sum_W n(\mathcal{X}, W)$ is the total count obtained by listing the $k$-element subsets $W$ and scoring 1 for each copy of $\mathcal{X}$ in $W$. The same number is found by taking each copy of $\mathcal{X}$ in $S$ and extending it in all possible ways to a set of size $k$, and this is given by $n(\mathcal{X}, S)\binom{v-p(\mathcal{X})}{k-p(\mathcal{X})}$. Therefore

$$I_k(S) = \binom{v}{k} + \sum_{\mathcal{X}} (-1)^{b(\mathcal{X})} n(\mathcal{X}, W) \binom{v - p(\mathcal{X})}{k - p(\mathcal{X})}.$$

□

If $k$ is small, the expression for $I_k(S)$ given by Theorem 4.3.1 only has a modest number of terms. Indeed, setting $k = 8$ and recalling from Table 4.1 the 31 configurations that have at most eight points,

$$\begin{aligned} I_8 \;=\; & \binom{v}{8} - a_0\binom{v-3}{5} + a_1\binom{v-6}{2} + a_2\binom{v-5}{3} \\ & - b_2 - (b_3+b_4)(v-7) - b_5\binom{v-6}{2} \\ & + c_{10} + c_{11} + c_{12} + (c_{14}+c_{15})(v-7) + p\binom{v-6}{2} \\ & - (m+d_2)(v-7) - d_3 - d_4 - d_5 - d_6 - d_7 \\ & + e_1(v-7) + e_2 + e_3 + e_6 + e_7 + e_8 \\ & - f_1(v-7) - f_2 - f_3 - f_{20} + g_1. \end{aligned}$$

After substituting from the formulae in section 4.2 and simplifying, this gives

$$\begin{aligned} I_8 \;=\; & n_v(v^5 - 80v^4 + 2575v^3 - 41820v^2 + 344724v - 1167600)/8! \\ & + p(v^2 - 37v + 354)/2 \\ & - (v-22)m + (v-25)e_1 + e_2 + e_3 \\ & - (v-28)f_1 - f_2 - f_3 + g_1. \end{aligned} \tag{4.3}$$

In a similar manner and with somewhat less effort we obtain formulae for $I_k$, $3 \leq k \leq 7$:

$$I_7 = \frac{n_v}{7!}(v^4 - 52v^3 + 1014v^2 - 8808v + 28905) + (v-15)p - m + e_1 - f_1,$$

$$I_6 = \frac{n_v}{6!}(v-9)(v-10)(v-12) + p, \quad I_5 = \frac{n_v}{120}(v-7)(v-9),$$

$$I_4 = \frac{n_v}{24}(v-6) \qquad \text{and} \qquad I_3 = \frac{n_v}{6}.$$

Before we end this section we mention that during the preparation of [24] it was observed that something similar to Theorem 4.3.1 could be proved for a much wider range of design-like constructs. A suitable generalization was published in [27]. It is also worth remarking that our particular form of this general theorem, namely our Theorem 4.3.1, is usable simply because for small $k$ it is actually feasible to construct all the configurations of at most $k$ points and compute the corresponding values of $n(\mathcal{X}, S)$.

## 4.4 STS(21)s

**Theorem 4.4.1** *Every Steiner triple system of order 21 has an independent set of cardinality eight.*

**Proof.** Let $S$ be a given STS(21). By (4.3), the number of occurrences in $S$ of eight independent points is

$$I_8(21) = 315 + 9p + m - 4e_1 + e_2 + e_3 + 7f_1 - f_2 - f_3 + g_1. \tag{4.4}$$

To deal with the terms in (4.4) with negative coefficients, we define three 9-point, 7-block configurations,

$$F_{37}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{2,4,8\},\{5,6,7\}\},$$

$$F_{39}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{2,5,7\},\{4,5,8\}\},$$

$$F_{44}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{4,6,8\},\{2,7,8\}\},$$

and use the corresponding subscripted lower-case letters $f_{37}$, $f_{39}$, $f_{44}$ to denote their frequencies of occurrence in $S$.

By considering the addition of a block to configuration $E_3$ linking one of these pairs of points of degree 2: {0, 5}, {0, 6}, {5, 6}, {1, 4}, {1, 7}, {4, 7}, thus converting the $E_3$ to either an $F_{44}$ or an $F_3$, we obtain the formula

$$6e_3 = 3f_3 + f_{44} \geq 3f_3.$$

Similarly, by adding a block to configuration $E_8$ linking points 0 and 7, or points 4 and 5, we obtain $e_8 = 3f_3 + f_{39}$, which, combined with the formula for $e_8$ from section 4.2, gives another upper bound for $f_3$, namely

$$6m = 3f_3 + f_{39} \geq 3f_3.$$

Finally, by adding a block to configuration $E_2$ linking points 2 and 4,

$$e_2 = 2f_2 + f_{37} \geq 2f_2.$$

We also mention that these three formulae can be obtained by application of Theorem 7.3.1. One can verify that the coefficients in each expression are in the same ratio as the orders of the full automorphism groups of the corresponding configurations. Indeed, $|\mathrm{Aut}(F_{37})| = |\mathrm{Aut}(F_{39})| = |\mathrm{Aut}(F_{44})| = 2$.

From the formulae for $d_2$ and $e_6$ from section 4.2 we obtain $e_1 \leq p$, and from those above, $f_3 \leq e_3 + m$ and $f_2 \leq \frac{1}{2}e_2$. Hence

$$I_8(21) \geq 315 + 5p + \frac{1}{2}e_2 + 7f_1 + g_1.$$

The proof of the theorem shows that, in fact, every STS(21) has at least 315 independent sets of cardinality eight. □

**Theorem 4.4.2** *Every Steiner triple system of order 21 is 4-colourable.*

**Proof.** By Theorem 4.4.1, every STS(21) has at least eight independent points. Given an STS(21), choose eight independent points and colour them red. Let $\mathcal{U}$ be the configuration consisting of the 13 points that are not coloured red and the 18 blocks that do not contain a red point. (There are 70 blocks altogether of which 28 contain two red points and 24 contain exactly one red point.) Denote by $p * q$ the third point in the block of the STS(21) that contains points $p$ and $q$.

Suppose there exists a point $x$ that occurs in exactly five blocks of $\mathcal{U}$,

$$\{x, a, b\}, \{x, c, d\}, \{x, e, f\}, \{x, g, h\}, \{x, i, j\}.$$

Let $k$ and $l$ be the remaining points of $\mathcal{U}$. We can assume that the points are labelled in such a way that $b*k$ is not equal to $l$, that $a*c$ is not equal to $e$ or $f$, and that $a*d$ is not equal to $e$ or $f$. A valid 4-colouring of the STS(21) is achieved by assigning colours as follows, yellow: $\{a, c, d, e, f\}$, blue: $\{g, h, i, j\}$ and green: $\{x, b, k, l\}$.

Alternatively, suppose $y$ is a point that occurs in only four blocks of $\mathcal{U}$,

$$\{y, a, b\}, \{y, c, d\}, \{y, e, f\}, \{y, g, h\},$$

with points $\{i, j, k, l\}$ of $\mathcal{U}$ remaining. We assume that the points are labelled such that $j * k$ is not equal to $l$, $a * i$ is not equal to $c$ or $d$, and $b * i$ is not equal to $c$ or $d$. Now colour $\{a, b, c, d, i\}$ yellow, $\{e, f, g, h\}$ blue and $\{y, j, k, l\}$ green.

Finally, a simple counting argument shows that there must exist a point of degree 4 or 5 in $\mathcal{U}$. Let $n$ be the number of points of degree 6 in $\mathcal{U}$. Then $n \leq 4$ since five such points would require at least $6 + 5 + 4 + 3 + 2 > 18$ blocks. Hence

$(3 \cdot 18 - 6n)/(13 - n) > 3$ and therefore it is impossible for all the remaining points to have degree less than 4 in $\mathcal{U}$. □

Hence $K(21) = \{3, 4\}$, the result promised in section 2.4.

## 4.5 Maximum independent sets

Let $S = (V, \mathcal{B})$ be a Steiner triple system of order $v$ and recall that a *maximum* independent set of $S$ is an independent set in $S$ of maximum cardinality. Recall also that if $I$ is a maximum independent set in $S$, then

$$|I| \leq \frac{v+\epsilon}{2}, \quad \text{where} \quad \begin{cases} \epsilon = 1 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ \epsilon = -1 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases} \tag{4.5}$$

In the case where $\epsilon = 1$, if $I$ has cardinality $(v+1)/2$, then the complement $V^* = V \backslash I$ together with $\mathcal{B}^*$, the blocks of $\mathcal{B}$ whose elements belong to $V^*$, is a Steiner triple system of order $(v-1)/2$, $(V^*, \mathcal{B}^*)$. Also we will use the fact that if $(V', \mathcal{B}')$ and $(V'', \mathcal{B}'')$ are two sub-STSs of $S$, then $(V' \cap V'', \{B \in \mathcal{B} : B \subseteq V' \cap V''\})$ is a sub-STS of $S$. In this section we prove two theorems concerning the existence of maximum independent sets in an STS($v$) where the upper bound (4.5) is achieved.

In what follows we shall often be concerned with subsets of $V$ and the various types of block that occur in $S$. It is convenient to define some notation. When we use upper case letters to denote subsets of $V$, we denote the cardinalities of points and blocks involving these subsets by corresponding lower case letters. Thus if $A, B \subseteq V$, we write $a = |A|$, $b = |B|$ and, for example, abb $= |\{\{\alpha, \beta, \gamma\} \in \mathcal{B} : \alpha \in A, \beta \in B, \gamma \in B\}|$, which we hope will not be confused with the product $a\,b^2$.

We are now ready to state and prove the theorems.

**Theorem 4.5.1** *Let $S = (V, \mathcal{B})$ be an STS($v$).*

*(i) If $v = 24w + 19$ or $24w + 27$, where $w \geq 0$, then $S$ cannot have more than one independent set of size $(v+1)/2$.*

*(ii) If $v = 24w + 7$ or $24w + 15$, where $w \geq 0$, then $S$ cannot have more than one independent set of size $(v+1)/2$ unless the point sets of the two complementary sub-STS($(v-1)/2$)s intersect in the point set of a sub-STS($(v-3)/4$). In this case the independent sets intersect in $(v+1)/4$ points.*

**Proof.** Suppose $S$ has two maximum independent sets, $I$ and $J$. Let $A = I \setminus J$, $B = J \setminus I$, $C = I \cap J$ and $D = V \setminus (I \cup J)$. Then $A \cup D$ is the point set of a sub-STS$(a+d)$, $B \cup D$ is the point set of a sub-STS$(b+d)$, and $D$, being the intersection of point sets of the Steiner triple systems on $A \cup D$ and $B \cup D$, is the point set of a sub-STS$(d)$. Furthermore, since all $DD$ pairs are contained in the blocks of the Steiner triple system on $D$, and $C$ is an independent set, it follows that $C \cup D$ is the point set of a sub-STS$(c+d)$. Also we have

$$a+b+c+d = v, \quad a = b, \quad a+c = (v+1)/2,$$

$$\mathrm{aaa} = \mathrm{aac} = \mathrm{acc} = \mathrm{bbb} = \mathrm{bbc} = \mathrm{bcc} = \mathrm{ccc} = 0,$$

$$\mathrm{ddd} = d(d-1)/6, \quad \mathrm{cdd} = 0, \quad \mathrm{ccd} = c(c-1)/2,$$

$$\mathrm{add} = \mathrm{bdd} = \mathrm{acd} = \mathrm{bcd} = 0, \quad \mathrm{abc} = a\,c,$$

$$\mathrm{aaa} + \mathrm{aab} + \mathrm{aac} + \mathrm{aad} = a(a-1)/2,$$

$$\mathrm{abb} + 3\,\mathrm{bbb} + \mathrm{bbc} + \mathrm{bbd} = b(b-1)/2,$$

$$2\,\mathrm{aab} + 2\,\mathrm{abb} + \mathrm{abc} + \mathrm{abd} = a\,b,$$

$$2\,\mathrm{aad} + \mathrm{abd} + \mathrm{acd} + 2\,\mathrm{add} = a\,d,$$

$$\mathrm{abd} + 2\,\mathrm{bbd} + \mathrm{bcd} + 2\,\mathrm{bdd} = b\,d.$$

Solving yields

$$\begin{aligned}
a = b &= \frac{1}{2}(-2\,d + v - 1), \ c = d+1, \\
\mathrm{aab} &= \frac{1}{12}\left(8\,d^2 + (10 - 6\,v)\,d + v^2 - 4\,v + 3\right), \\
\mathrm{aad} &= \frac{1}{24}\left(-4\,d^2 + 4\,d + v^2 - 4\,v + 3\right), \\
\mathrm{abb} &= \frac{1}{12}\left(8\,d^2 + (10 - 6\,v)\,d + v^2 - 4\,v + 3\right), \\
\mathrm{bbd} &= \frac{1}{24}\left(-4\,d^2 + 4\,d + v^2 - 4\,v + 3\right), \\
\mathrm{abd} &= \frac{1}{12}\left(-8\,d^2 + 6\,v\,d - 10\,d - v^2 + 4\,v - 3\right).
\end{aligned}$$

Observing that $\mathrm{aab} = -\mathrm{abd}$, it follows that $\mathrm{aab} = \mathrm{abd} = 0$, and solving the quadratic $\mathrm{aab} = 0$ yields $d = (v-3)/4$ or $d = (v-1)/2$. Since $d = (v-1)/2$ implies $a = b = 0$, it must be that $d = (v-3)/4$.

If $v = 24w + 19$ or $24w + 27$, $w \geq 0$, then $d$ is not admissible. This proves (i). On the other hand, if $v = 24w + 7$ or $24w + 15$, $w \geq 0$, then $d$ is admissible, $c = d + 1 = (v + 1)/4$ and the proof of (ii) is complete. □

**Theorem 4.5.2** *Let $S = (V, \mathcal{B})$ be an STS$(v)$ with two distinct independent sets, $I$ and $J$, of size $(v-1)/2$.*

*(i) Suppose $v = 12w + 21$ or $12w + 25$, where $w \geq 0$. Then $I$ and $J$ intersect in $(v-1)/4-2$, $(v-1)/4-1$ or $(v-1)/4$ points. Furthermore, if the intersection has $(v-1)/4-2$ points, then $I \cap J \cup (V \setminus (I \cup J))$ is the point set of a sub-STS$((v-1)/2-3)$.*

*(ii) Suppose $v = 12w + 15$ or $12w + 19$, where $w \geq 0$. Then $I$ and $J$ intersect in $(v+1)/4-2$, $(v+1)/4-1$, $(v+1)/4$ or $(v-3)/2$ points.*

**Proof.** As before, let $A = I \setminus J$, $B = J \setminus I$, $C = I \cap J$ and $D = V \setminus (I \cup J)$. The equations to solve are

$$a + b + c + d = v, \;\; a = b, \;\; a + c = (v-1)/2,$$

$$\mathrm{aaa} = \mathrm{aac} = \mathrm{acc} = \mathrm{bbb} = \mathrm{bbc} = \mathrm{bcc} = \mathrm{ccc} = 0,$$

$$3\,\mathrm{aaa} + \mathrm{aab} + \mathrm{aac} + \mathrm{aad} = a(a-1)/2,$$

$$\mathrm{abb} + 3\,\mathrm{bbb} + \mathrm{bbc} + \mathrm{bbd} = b(b-1)/2,$$

$$\mathrm{acc} + \mathrm{bcc} + 3\,\mathrm{ccc} + \mathrm{ccd} = c(c-1)/2,$$

$$\mathrm{add} + \mathrm{bdd} + \mathrm{cdd} + 3\,\mathrm{ddd} = d(d-1)/2,$$

$$2\,\mathrm{aab} + 2\,\mathrm{abb} + \mathrm{abc} + \mathrm{abd} = a\,b,$$

$$2\,\mathrm{aac} + \mathrm{abc} + 2\,\mathrm{acc} + \mathrm{acd} = a\,c,$$

$$2\,\mathrm{aad} + \mathrm{abd} + \mathrm{acd} + 2\,\mathrm{add} = a\,d,$$

$$\mathrm{abc} + 2\,\mathrm{bbc} + 2\,\mathrm{bcc} + \mathrm{bcd} = b\,c,$$

$$\mathrm{abd} + 2\,\mathrm{bbd} + \mathrm{bcd} + 2\,\mathrm{bdd} = b\,d,$$

$$\mathrm{acd} + \mathrm{bcd} + 2\,\mathrm{ccd} + 2\,\mathrm{cdd} = c\,d,$$

and their solution with parameters add, bdd and cdd is

$$\begin{aligned}
a = b &= \frac{1}{2}(-2\,d+v+1),\ c=d-1,\\
\mathrm{aab} &= \frac{1}{12}(8\,d^2-6\,v\,d-2\,d+v^2+8\,\mathrm{add}-4\,\mathrm{bdd}-4\,\mathrm{cdd}+2\,v-3),\\
\mathrm{aad} &= \frac{1}{24}(-4\,d^2+4\,d+v^2-16\,\mathrm{add}+8\,\mathrm{bdd}+8\,\mathrm{cdd}-4\,v+3),\\
\mathrm{abb} &= \frac{1}{12}(8\,d^2-6\,v\,d-2\,d+v^2-4\,\mathrm{add}+8\,\mathrm{bdd}-4\,\mathrm{cdd}+2\,v-3),\\
\mathrm{abc} &= \mathrm{cdd}-\frac{1}{2}(d-1)\,(2\,d-v+1),\\
\mathrm{abd} &= \frac{1}{12}(-8\,d^2+6\,v\,d-10\,d-v^2-8\,\mathrm{add}-8\,\mathrm{bdd}+4\,\mathrm{cdd}+4\,v+9),\\
\mathrm{acd} &= -\mathrm{cdd}+d-1,\\
\mathrm{bbd} &= \frac{1}{24}(-4\,d^2+4\,d+v^2+8\,\mathrm{add}-16\,\mathrm{bdd}+8\,\mathrm{cdd}-4\,v+3),\\
\mathrm{bcd} &= -\mathrm{cdd}+d-1,\\
\mathrm{ccd} &= \frac{1}{2}(d-2)\,(d-1),\\
\mathrm{ddd} &= \frac{1}{6}(d^2-d-2\,\mathrm{add}-2\,\mathrm{bdd}-2\,\mathrm{cdd}).
\end{aligned}$$

First we deal with (i). If $d=(v-1)/4-2-x$ and $x\geq 0$, then

$$\mathrm{abd}+\mathrm{acd}=\frac{1}{6}\big(-4\,x^2-v\,x-19\,x-4\,\mathrm{add}-4\,\mathrm{bdd}-4\,\mathrm{cdd}-24\big),$$

which is clearly negative. Hence $d\geq(v-1)/4-1$. Next, we have

$$\mathrm{aab}+2\,\mathrm{abb}+\mathrm{abd}+\mathrm{cdd}=\frac{1}{4}\big(8\,d^2-6\,(v+1)\,d+v^2+4\,v-1\big).$$

As a quadratic in $d$ the right hand side has roots

$$d=\frac{1}{8}\big(3\,v+3\pm\sqrt{v^2-14\,v+17}\big).$$

Since we are assuming that $v\geq 21$, we have $v^2-14\,v+17>(v-9)^2$, and we can deduce from the smaller root that $d\leq(v-1)/4+1$, or, by considering the larger root, that $d\geq(v-1)/2$. To eliminate this last possibility we argue as follows.

If $d>(v-1)/2$, then $A$ and $B$ are empty. Suppose $d=(v-1)/2$ and $v\geq 21$. Then $a=b=1$ and $\mathrm{aab}=\mathrm{abb}=0$. Hence $\mathrm{add}=\mathrm{bdd}=\mathrm{cdd}$ and $\mathrm{cdd}=0$ or 1. In the first case, $\mathrm{ddd}=\big(v^2-4\,v+3\big)/24$; in the second case, $\mathrm{ddd}=\big(v^2-4\,v-21\big)/24$. In both cases $\mathrm{ddd}>d(d-2)/6=(v-5)\,(v-1)/24$, which, since $d$ is even, means that there is no possibility of the ddd blocks forming a valid configuration.

Thus we have shown that $(v-1)/4-2 \leq c = d-1 \leq (v-1)/4$.

Now suppose $c = (v-1)/4-2$. Then $d = (v-1)/4-1$ and $3\,\mathrm{abd} + \mathrm{acd} = -2(\mathrm{add} + \mathrm{bdd})$; hence $\mathrm{add} = \mathrm{bdd} = 0$. After substituting these values in the expressions for abd and acd we obtain $\mathrm{abd} = -\mathrm{acd} = (9 + 4\mathrm{cdd} - v)/12$, which implies that $\mathrm{abd} = \mathrm{acd} = 0$ and $\mathrm{cdd} = (v-9)/4$. Hence

$$\mathrm{ccc} + \mathrm{ccd} + \mathrm{cdd} + \mathrm{ddd} = \frac{1}{24}(v-9)(v-7),$$

which is the correct number of blocks for an $\mathrm{STS}((v-1)/2-3)$ with point set $C \cup D$. This completes the proof of (i).

Now we consider (ii). The details are similar. If $d = (v+1)/4-2-x$ and $x \geq 0$, then

$$3\,\mathrm{abd} + \mathrm{acd} = \frac{1}{4}(-8\,\mathrm{add} - 8\,\mathrm{bdd} - (2\,x+1)(v+4\,x+9)),$$

which is clearly negative. Hence $d \geq (v+1)/4-1$.

As before,

$$\mathrm{aab} + 2\,\mathrm{abb} + \mathrm{abd} + \mathrm{cdd} = \frac{1}{4}\left(8\,d^2 - 6\,(v+1)\,d + v^2 + 4\,v - 1\right),$$

and the roots of the right hand side are

$$d = \frac{1}{8}\left(3\,v + 3 \pm \sqrt{v^2 - 14\,v + 17}\right).$$

Since $v \geq 15$, we have $v^2 - 14\,v + 17 > (v-10)^2$ and we can deduce that $d \leq (v+1)/4+1$ or $d = (v-1)/2$.

Finally, we recall that $|I \cap J| = c = d-1$; hence $(v+1)/4-1 \leq c \leq (v+1)/4$ or $c = (v-3)/2$. □

# Chapter 5

# Trades in Steiner triple systems

## 5.1 Introduction

This chapter is primarily concerned with trades in Steiner triple systems. A *trade set*, or *n-way trade*, $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, $n \geq 2$, is a set of pairwise disjoint $m$-block configurations, $T_i$, which has the property that every pair of distinct elements occurs in precisely the same number (zero or one) of blocks of each $T_i$. Traditionally, the number of blocks is called the *volume* of the trade set, denoted by $\text{vol}(\mathcal{T})$, and the *foundation* of the trade set, $\text{found}(\mathcal{T})$, is the set of elements covered by each $T_i$. A 2-way trade $\{T_1, T_2\}$ is simply called a *trade*. We shall make the distinction between a trade $\mathcal{T}$ and its constituent configurations $T_i$; these configurations will be called *tradeable configurations*.

**Definition 5.1.1** Two trade sets $\mathcal{T} = \{T_1, T_2, \ldots, T_{n_1}\}$ and $\mathcal{T}' = \{T'_1, T'_2, \ldots, T'_{n_2}\}$ are said to be *isomorphic* if

(i) $\text{vol}(\mathcal{T}) = \text{vol}(\mathcal{T}')$,

(ii) $\text{found}(\mathcal{T}) = \text{found}(\mathcal{T}')$,

(iii) $n_1 = n_2 = n$, and

(iv) there exists a function $f : \text{found}(\mathcal{T}) \to \text{found}(\mathcal{T}')$ such that

$$f(\{T_1, T_2, \ldots, T_n\}) = \{T'_1, T'_2, \ldots, T'_n\}.$$

In section 5.3, we enumerate the non-isomorphic tradeable configurations and trade sets with volume at most 12 that can occur in Steiner triple systems. Observe that an $n$-way trade gives rise to $\binom{n}{l}$ $l$-way trades for $2 \leq l \leq n$. However, some of these $l$-way trades may be isomorphic, and our computational results reflect this.

We partly replicate some of the results in Khosrovshahi & Maimani [44], where the numbers of such trades of volume at most nine are given.

## 5.2 Algorithms

We begin this section with a few remarks about labellings.

Following Colbourn & Rosa [16], we extend the usual ordering $<$ of the integers to pairs of integers, triples and sets of triples. Pairs of integers are given the reverse-lexicographical ordering; for $a < b$ and $c < d$, $\{a, b\} < \{c, d\}$ if $b < d$, or $b = d$ and $a < c$. Triples are ordered by their smallest pairs; if $a < b < c$ and $d < e < f$, then $\{a, b, c\} < \{d, e, f\}$ if $\{a, b\} < \{d, e\}$, or $\{a, b\} = \{d, e\}$ and $c < f$. For two sets of triples, $A$ and $B$, $A < B$ if the smallest triple in $A \setminus B$ is less than the smallest triple in $B \setminus A$.

A *labelling* of a configuration $C$ with point set $V$ is a function $\phi$ which maps $V$ onto the set $\{0, 1, \ldots, |V| - 1\}$ of *labels*. A *canonical labelling* of a configuration $C$ is a labelling $\phi$ for which $\phi(C)$ is as small as possible. If two configurations have the same canonical labellings, they are isomorphic. The number of canonical labellings of $C$ is equal to the order of $\mathrm{Aut}(C)$, the full automorphism group of $C$.

It is relatively straightforward to determine all trade sets $\mathcal{T}$ with $\mathrm{vol}(\mathcal{T}) \leq 7$ by elementary arguments, but after that an electronic computation is desirable. Our main algorithm uses a simple back-tracking procedure to generate all possible labelled trades $\{C, D\}$ from a given labelled configuration $C$. It is clear from the following presentation that it has the desired effect.

**Algorithm 5.2.1** Suppose we are given a labelled configuration $C$ with point set $V$. Set $L = \{\{r, s\} : \{r, s\}$ is a pair of points in $C\}$ and set $D = \{\}$. Then we perform a procedure called ADD BLOCK to add triples one at a time to $D$.

ADD BLOCK

Choose a pair $\{r, s\}$ in $L$.

For each point $t \in V \setminus \{r, s\}$ for which $\{r, s, t\}$ is not a block in $C$ and both $\{r, t\}$ and $\{s, t\}$ are pairs in $L$:

Add $\{r, s, t\}$ to $D$ and remove the pairs $\{r, s\}$, $\{r, t\}$, $\{s, t\}$ from $L$.

If $L$ is empty, report a trade, $\{C, D\}$; otherwise perform the procedure ADD BLOCK.

Remove $\{r, s, t\}$ from $D$ and restore its pairs to $L$.

Return.

Algorithm 5.2.1 requires a list containing every tradeable configuration of $n$ blocks. Therefore we first implemented an algorithm to determine, for $1 \leq n \leq 10$, all pairwise non-isomorphic $n$-block configurations which can occur in a Steiner triple system. The basic method is straightforward. We add a new triple in every possible manner to every $(n-1)$-block configuration and reject isomorphs using Miller's algorithm [55]. As described in section 4.2 of Colbourn & Rosa [16], Miller's algorithm constructs canonical labellings for Steiner triple systems. For efficiency we found it was necessary to implement a number of elementary enhancements to make the algorithm work efficiently for configurations that are not complete designs. We describe some of these improvements here.

(i) Suppose $\phi$ is a partial labelling of the point set $V$ of a configuration. Let $V_0$ denote the set of unlabelled points and let

$$\begin{aligned} V_1 &= \{x \in V_0 : x \text{ occurs in the same block as a labelled point}\}, \\ V_2 &= \{x \in V_0 \setminus V_1\}. \end{aligned}$$

Suppose $V_1 \cap V_2$ is non-empty. Then extending $\phi$ by assigning the smallest unused label to a point in $V_2$ will not produce a canonical labelling.

(ii) Let $C$ be a configuration of $v$ points. Let $D = \{\{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}, \ldots, \{a_k, b_k, c_k\}\}$ be the set of blocks of $C$ that have three points of degree 1. Then it suffices to canonically label the points of $C \setminus D$ first and then label the points of $D$ in the order $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$, $\ldots$, $a_k$, $b_k$, $c_k$. Also $\mathrm{Aut}(C) = 2^k(3k)!!!\ |\mathrm{Aut}(C \setminus D)|$.

(iii) Let $C$ be a configuration and let $S$ be a set of blocks of $C$ of the form $\{\{x, a_1, b_1\}, \{x, a_2, b_2\}, \ldots \{x, a_k, b_k\}\}$, where, $k > 1$, the points $a_1$, $b_1$, $a_2$, $b_2$, $\ldots$, $a_k$,

$b_k$ have degree 1, and for given $x$, $k$ is as large as possible. Then in any canonical labelling of $C$, the points $x$, $a_1$, $b_1$, $a_2$, $b_2$, $\ldots$, $a_k$, $b_k$ are labelled consecutively. Also $\mathrm{Aut}(C) = (2k)!!\ |\mathrm{Aut}(C \setminus S)|$.

(iv) Let $\phi_0$ be a (not necessarily canonical) labelling of a configuration $C$ and let $\phi$ be a partial labelling of $C$ with the property that no block of $C$ contains precisely two unlabelled points. Denote the inverses of $\phi_0$ and $\phi$ by $\phi'_0$ and $\phi'$, respectively. Suppose that $\phi(\phi'(i_0) * \phi'(j_0)) > \phi_0(\phi'_0(i_0) * \phi'_0(j_0))$ for some pair $\{i_0, j_0\}$ of labels used by $\phi$ and that $\phi(\phi'(i) * \phi'(j)) = \phi_0(\phi'_0(i) * \phi'_0(j))$ for all pairs of labels $\{i, j\} < \{i_0, j_0\}$. Then $\phi$ cannot be extended to a canonical labelling.

The numbers, $C(n)$, of $n$-block configurations are known for $1 \leq n \leq 8$ (Grannell & Griggs [33]) and so provide a check on the correctness of the program. Also we would like to thank Professor C. J. Colbourn, who kindly made available to us a list of all 6-, 7- and 8-block configurations thereby providing independent verification of our computations for $6 \leq n \leq 8$.

From our collection of ($\leq 10$)-block configurations we constructed for $n = 11, 12$ all pairwise non-isomorphic $n$-block configurations in which each point has degree at least 2. Observe that any 10-block subconfiguration of an $n$-block configuration with no points of degree 1 has at most $3(n-10)$ points of degree 1. Hence we can begin with all 10-block configurations containing at most $3(n-10)$ points of degree 1 and extend them in every possible way so that every point has degree at least 2, rejecting isomorphs with Miller's algorithm.

We denote the number of configurations where every point has degree $\geq 2$ by $B(n)$. There are two reasons why it is relevant to identify configurations all of whose points have degree at least 2. First, it is easy to see that every point of a tradeable configuration must have degree greater than 1. Secondly, frequency counts in Steiner triple systems for ($\leq n$)-block configurations in which every point has degree at least 2 in some sense generate the frequency counts for all ($\leq n$)-block configurations, as explained in Chapter 4.

Thus for $1 \leq n \leq 12$ we were able to construct a list containing every tradeable configuration of $n$ blocks. By giving the points of these configurations a canonical labelling and then applying Algorithm 5.2.1 we were therefore able to create a list of the different labelled trades $\{C, D\}$ that originate from each canonically labelled tradeable configuration $C$.

Before moving on to the next section we describe a second procedure for generating tradeable configurations. This method also uses Algorithm 5.2.1 but it does not require an initial set of configurations.

**Algorithm 5.2.2** [Given $n$, generate all pairwise non-isomorphic tradeable configurations of $n$ blocks.] Set $C = D = \{\}$ and $v = 0$. Then perform procedure ADD BLOCK, which recursively adds triples alternately to $C$ and $D$. During the execution of this algorithm the points of $C \cup D$ are $\{0, 1, 2, \ldots, v-1\}$.

ADD BLOCK

If $C$ and $D$ contain precisely the same pairs:

If $C$ has exactly $n$ blocks, report that $(C, D)$ is a trade and return.

[Add a block to $C$ containing three new pairs.] For each triple $\{r, s, t\}$, where $0 \leq r < s < t \leq v + 2$, if none of the pairs $\{r, s\}$, $\{r, t\}$ and $\{s, t\}$ are in $C$:

Add $\{r, s, t\}$ to $C$.

Replace $v$ by $\max\{v, t + 1\}$.

Perform procedure ADD BLOCK.

Remove $\{r, s, t\}$ from $C$ and restore $v$.

Return.

If $C$ has the same number of blocks as $D$ (but different sets of pairs):

If $C$ has $n$ blocks, return.

[Add a block to $C$.] Choose a pair $\{r, s\}$ in $D$ but not in $C$. For each point $t$, $0 \leq t \leq v$, apart from $r$ and $s$, if neither $\{r, t\}$ nor $\{s, t\}$ is a pair in $C$ and if $\{r, s, t\}$ is not a block in $D$:

Add $\{r, s, t\}$ to $C$.

If $t = v$, replace $v$ by $v + 1$.

Perform procedure ADD BLOCK.

Remove $\{r, s, t\}$ from $C$ and restore $v$.

Return.

If $C$ has more blocks than $D$:

[Add a block to $D$.] Choose a pair $\{r, s\}$ in $C$ but not in $D$. For each point $t$, $0 \leq t \leq v$, apart from $r$ and $s$, if neither $\{r, t\}$ nor $\{s, t\}$ is a pair in $D$ and if $\{r, s, t\}$ is not a block in $C$:

Add $\{r, s, t\}$ to $D$.

If $t = v$, replace $v$ by $v + 1$.

Perform procedure ADD BLOCK.

Remove $\{r, s, t\}$ from $D$ and restore $v$.

Return.

We run Algorithm 5.2.2 to generate trades $(C, D)$ of a given volume starting from nothing. Then we extract the corresponding tradeable configurations, canonically label them and reject duplicates. Algorithm 5.2.2 does not necessarily generate all *labelled* trades of volume $n$. However, amongst the trades that it does generate are all pairwise non-isomorphic $n$-block tradeable configurations. For efficiency, it is preferable that throughout the execution of the algorithm the size of the point set should be as small as possible subject to the condition that the trades $(C, D)$ generated include (as $C$) all pairwise non-isomorphic tradeable configurations. This condition is satisfied if, assuming that there are $v$ points currently in the system, whenever a block is added, either (i) if the block contains three new pairs, its points are chosen from $\{0, 1, \ldots, v + 2\}$, or (ii) if the block contains a pair already in the system, the third point is chosen from $0, 1, \ldots, v$.

In fact we employed both methods to generate tradeable configurations (with consistent results!) and for $n = 12$ we found that using Algorithm 5.2.2 was considerably less demanding of computer resources than the creation of all 10,695,820 12-block configurations having no points of degree 1.

## 5.3 Results

First, we give in Table 5.3.1 the number, $C(n)$, of $n$-block configurations, $1 \leq n \leq 10$, the number, $B(n)$, of $n$-block configurations in which each point has degree at least 2, $1 \leq n \leq 12$ and the number, $A(n)$, of $n$-block tradeable configurations, $1 \leq n \leq 12$.

Listings of all $n$-block configurations, together with formulae for their numbers of occurrence in terms of $v$ (the order of the Steiner triple system), $p$ (the number of Pasch configurations), and $m$ (the number of mitres) are given in Grannell, Griggs & Mendelsohn [34] for $1 \leq n \leq 4$ and Danziger, Mendelsohn, Grannell & Griggs [17] for $n = 5$.

Table 5.3.1. Configuration counts

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $C(n)$ | 1 | 2 | 5 | 16 | 56 | 282 | 1865 | 17100 |
| $B(n)$ | 0 | 0 | 0 | 1 | 1 | 5 | 19 | 153 |
| $A(n)$ | 0 | 0 | 0 | 1 | 0 | 2 | 2 | 10 |

| $n$ | 9 | 10 | 11 | 12 |
|---|---|---|---|---|
| $C(n)$ | 207697 | 3180571 | - | - |
| $B(n)$ | 1615 | 25180 | 479238 | 10695820 |
| $A(n)$ | 17 | 102 | 436 | 3822 |

The $n$-block configurations in which each point has degree at least 2, $1 \leq n \leq 7$, are listed in Table 5.3.2. Set brackets and delimiting commas are omitted. The configurations have canonical labellings with the blocks presented in lexicographical order.

In Table 5.3.3 $A(n, m)$ is the number of $n$-block, $m$-point tradeable configurations. We denote by $L(n, m)$ the number of labelled (2-way) trades of volume $n$ and foundation $m$ counted as follows. We take each canonically labelled tradeable configuration $C$ of $n$ blocks and $m$ points and count every possible (2-way) trade between $C$ and labelled configurations $D$. These are precisely the trades that are generated by Algorithm 5.2.1. They are not necessarily pairwise non-isomorphic.

At the end of this chapter, we present a table of the 124 pairwise non-isomorphic trade sets of volume up to and including 10 (Table 5.3.4). The trades are arranged

| Table 5.3.2. Configurations with no points of degree 1 | | | | | |
|---|---|---|---|---|---|
| | Blocks | Points | | | \|Aut\| |
| 1 | 4 | 6 | 012 034 135 245 | $C_{16}$ Pasch | 24 |
| 2 | 5 | 7 | 012 034 135 236 456 | $D_1$ mitre | 12 |
| 3 | 6 | 7 | 012 034 135 146 236 245 | $E_1$ semihead | 24 |
| 4 | 6 | 8 | 012 034 135 147 236 567 | $E_2$ | 2 |
| 5 | 6 | 8 | 012 034 135 246 257 367 | $E_3$ 6-cycle | 12 |
| 6 | 6 | 9 | 012 034 135 267 468 578 | $E_4$ | 12 |
| 7 | 6 | 9 | 012 034 156 278 357 468 | $E_5$ | 72 |
| 8 | 7 | 7 | 012 034 056 135 146 236 245 | $F_1$ STS(7) | 168 |
| 9 | 7 | 8 | 012 034 057 135 146 236 247 | $F_2$ | 4 |
| 10 | 7 | 8 | 012 034 135 147 236 257 456 | $F_3$ | 6 |
| 11 | 7 | 9 | 012 034 058 135 147 236 678 | $F_4$ | 6 |
| 12 | 7 | 9 | 012 034 135 147 168 236 578 | $F_5$ | 2 |
| 13 | 7 | 9 | 012 034 135 147 236 258 678 | $F_6$ | 1 |
| 14 | 7 | 9 | 012 034 135 147 236 468 578 | $F_7$ | 1 |
| 15 | 7 | 9 | 012 034 078 135 236 457 468 | $F_8$ | 4 |
| 16 | 7 | 9 | 012 034 135 178 236 457 468 | $F_9$ | 1 |
| 17 | 7 | 9 | 012 034 067 135 168 245 278 | $F_{10}$ | 12 |
| 18 | 7 | 9 | 012 034 067 135 168 245 378 | $F_{11}$ | 6 |
| 19 | 7 | 9 | 012 034 067 135 168 245 478 | $F_{12}$ | 2 |
| 20 | 7 | 9 | 012 034 078 135 168 246 257 | $F_{13}$ | 12 |
| 21 | 7 | 9 | 012 034 135 168 246 257 378 | $F_{14}$ | 2 |
| 22 | 7 | 10 | 012 034 067 135 268 479 589 | $F_{15}$ | 6 |
| 23 | 7 | 10 | 012 034 067 135 268 489 579 | $F_{16}$ | 2 |
| 24 | 7 | 10 | 012 034 135 236 478 579 689 | $F_{17}$ | 6 |
| 25 | 7 | 10 | 012 034 078 135 246 579 689 | $F_{18}$ | 4 |
| 26 | 7 | 10 | 012 034 135 178 246 579 689 | $F_{19}$ | 2 |

by volume and then by foundation size. The points are labelled with non-negative integers; set brackets and commas have been omitted and labels 10, 11, ... are represented by lower case roman letters a, b, ... respectively. Each non-isomorphic tradeable configuration is assigned a unique number in this table, thus making it easy to distinguish between trades both of whose configurations are isomorphic and trades where the configurations are non-isomorphic.

The 124 trade sets consist of 117 2-way trade sets, six 3-way trade sets and one 4-way trade set. There is a small amount of duplication; for $k > 2$ the pairwise non-isomorphic $(k-1)$-way subsets of a $k$-way trade set appear as separate entries in the list. But this seems the clearest way to present the results. Thus the (2-way) trades numbered 2, 22, 24 and 27 in the list are sub-trades of the 3-way trade sets numbered 3, 23, 25 and 28, respectively, and the (2-way) trade number 7 is a sub-

| Table 5.3.3. Counts of tradeable configurations and trade sets | | | | | | | |
|---|---|---|---|---|---|---|---|
| Blocks | Points | | | Non-isomorphic trades | | | |
| $n$ | $m$ | $A(n,m)$ | $L(n,m)$ | 2-way | 3-way | 4-way | |
| 4 | 6 | 1 | 1 | 1 | | | Pasch |
| 6 | 7 | 1 | 2 | 1 | 1 | | semihead |
| 6 | 8 | 1 | 1 | 1 | | | 6-cycle |
| 7 | 7 | 1 | 8 | 1 | | | STS(7) |
| 7 | 9 | 1 | 1 | 1 | | | |
| 8 | 8 | 1 | 3 | 1 | 1 | 1 | (1) |
| 8 | 9 | 3 | 3 | 3 | | | |
| 8 | 10 | 4 | 4 | 4 | | | |
| 8 | 11 | 1 | 1 | 1 | | | (2) |
| 8 | 12 | 1 | 1 | 1 | | | (3) |
| 9 | 9 | 7 | 11 | 7 | 3 | | |
| 9 | 10 | 7 | 8 | 5 | | | |
| 9 | 11 | 3 | 3 | 2 | | | |
| 10 | 9 | 3 | 12 | 3 | 1 | | |
| 10 | 10 | 37 | 51 | 29 | | | |
| 10 | 11 | 39 | 43 | 34 | | | |
| 10 | 12 | 19 | 21 | 18 | | | |
| 10 | 13 | 3 | 4 | 3 | | | |
| 10 | 14 | 1 | 1 | 1 | | | (4) |
| | | 134 | 179 | 117 | 6 | 1 | |

(1) point-deleted STS(9)
(2) two Pasch configurations with a common point
(3) two disjoint Pasch configurations
(4) disjoint Pasch configuration and 6-cycle

trade of the 3-way trade number 8 which in turn is a sub-trade of the 4-way trade number 9. In Table 5.3.4 the first configuration in each trade set has its canonical labelling with the blocks presented in lexicographical order.

There are 89 trade sets of volume 10, all but one of which are 2-way. Of these, 72 trade sets are between isomorphic configurations and 16 between non-isomorphic configurations. But the 3-way trade set in row 38 is of particular interest. The three tradeable configurations are:

| | | |
|---|---|---|
| (i) | 012 034 057 068 135 146 178 236 247 258 | (Configuration 33) |
| (ii) | 013 026 047 058 124 157 168 235 278 346 | (Configuration 34) |
| (iii) | 018 027 035 046 125 136 147 234 268 578 | (Configuration 34) |

Configurations (ii) and (iii) are isomorphic but are not isomorphic to configuration (i). The 3-way trade set gives rise to two 2-way trade sets, one between isomorphic tradeable configurations ((ii) and (iii)) and one between non-isomorphic tradeable configurations ((i) and (ii)). Another interesting situation occurs in rows 45 and 46 with the three following non-isomorphic tradeable configurations.

| | | |
|---|---|---|
| (iv) | 012 034 057 068 135 146 236 245 569 789 | (Configuration 45) |
| (v) | 013 025 046 078 126 145 234 356 579 689 | (Configuration 46) |
| (vi) | 012 034 056 078 135 146 179 236 245 289 | (Configuration 44) |

There are 2-way trade sets between configurations (iv) and (v) and between (iv) and (vi) but not between configurations (v) and (vi) because of the common block $\{0, 7, 8\}$. Hence the 2-way trades sets do not extend to a 3-way trade set.

| Table 5.3.4. Pairwise non-isomorphic trade sets | | | | |
|---|---|---|---|---|
| | Blocks | Points | Config. | |
| 1 | 4 | 6 | 1 | 012 034 135 245 |
| | | | 1 | 013 024 125 345 |
| 2 | 6 | 7 | 2 | 012 034 135 146 236 245 |
| | | | 2 | 013 024 126 145 235 346 |
| 3 | 6 | 7 | 2 | 012 034 135 146 236 245 |
| | | | 2 | 013 024 126 145 235 346 |
| | | | 2 | 014 023 125 136 246 345 |
| 4 | 6 | 8 | 3 | 012 034 135 246 257 367 |
| | | | 3 | 013 024 125 267 346 357 |
| 5 | 7 | 7 | 4 | 012 034 056 135 146 236 245 |
| | | | 4 | 013 025 046 126 145 234 356 |
| 6 | 7 | 9 | 5 | 012 034 067 135 168 245 378 |
| | | | 5 | 016 024 037 125 138 345 678 |
| 7 | 8 | 8 | 6 | 012 034 067 135 147 236 257 456 |
| | | | 6 | 013 026 047 127 145 235 346 567 |
| 8 | 8 | 8 | 6 | 012 034 067 135 147 236 257 456 |
| | | | 6 | 013 026 047 127 145 235 346 567 |
| | | | 6 | 014 027 036 123 157 256 345 467 |
| 9 | 8 | 8 | 6 | 012 034 067 135 147 236 257 456 |
| | | | 6 | 013 026 047 127 145 235 346 567 |
| | | | 6 | 014 027 036 123 157 256 345 467 |
| | | | 6 | 017 023 046 125 134 267 356 457 |
| 10 | 8 | 9 | 7 | 012 034 057 135 146 236 278 568 |
| | | | 7 | 014 027 035 123 156 268 346 578 |
| 11 | 8 | 9 | 8 | 012 034 135 146 178 236 247 258 |
| | | | 8 | 013 024 126 147 158 235 278 346 |
| 12 | 8 | 9 | 9 | 012 034 135 147 236 258 378 468 |
| | | | 9 | 014 023 125 137 268 346 358 478 |
| 13 | 8 | 10 | 10 | 012 034 135 146 178 236 379 589 |
| | | | 10 | 014 023 126 137 158 346 359 789 |
| 14 | 8 | 10 | 11 | 012 034 067 089 135 245 568 579 |
| | | | 11 | 013 024 068 079 125 345 567 589 |
| 15 | 8 | 10 | 12 | 012 034 135 246 257 289 368 379 |
| | | | 12 | 013 024 125 268 279 346 357 389 |
| 16 | 8 | 10 | 13 | 012 034 135 246 257 368 589 679 |
| | | | 13 | 013 024 125 267 346 358 579 689 |
| 17 | 8 | 11 | 14 | 012 034 067 089 135 245 68a 79a |
| | | | 14 | 013 024 068 079 125 345 67a 89a |
| 18 | 8 | 12 | 15 | 012 034 135 245 678 69a 79b 8ab |
| | | | 15 | 013 024 125 345 679 68a 78b 9ab |

| | Blocks | Points | Config. | |
|---|---|---|---|---|
| 19 | 9 | 9 | 16 | 012 034 057 068 135 146 178 236 245 |
| | | | 16 | 013 026 045 078 124 157 168 235 346 |
| 20 | 9 | 9 | 17 | 012 034 057 135 146 236 247 258 378 |
| | | | 17 | 014 027 035 125 136 238 246 347 578 |
| 21 | 9 | 9 | 18 | 012 034 057 135 146 236 247 258 678 |
| | | | 18 | 014 027 035 125 136 234 268 467 578 |
| 22 | 9 | 9 | 19 | 012 034 058 135 147 236 278 468 567 |
| | | | 19 | 014 028 035 123 157 267 346 478 568 |
| 23 | 9 | 9 | 19 | 012 034 058 135 147 236 278 468 567 |
| | | | 19 | 014 028 035 123 157 267 346 478 568 |
| | | | 19 | 015 023 048 127 134 268 356 467 578 |
| 24 | 9 | 9 | 20 | 012 034 058 135 147 236 248 257 456 |
| | | | 20 | 014 028 035 123 157 247 256 346 458 |
| 25 | 9 | 9 | 20 | 012 034 058 135 147 236 248 257 456 |
| | | | 20 | 014 028 035 123 157 247 256 346 458 |
| | | | 20 | 015 023 048 127 134 246 258 356 457 |
| 26 | 9 | 9 | 21 | 012 034 078 135 147 236 258 468 567 |
| | | | 21 | 017 023 048 125 134 268 356 467 578 |
| 27 | 9 | 9 | 22 | 012 034 078 135 168 246 257 367 458 |
| | | | 22 | 013 027 048 126 158 245 346 357 678 |
| 28 | 9 | 9 | 22 | 012 034 078 135 168 246 257 367 458 |
| | | | 22 | 013 027 048 126 158 245 346 357 678 |
| | | | 22 | 018 024 037 125 136 267 345 468 578 |
| 29 | 9 | 10 | 23 | 012 034 057 135 146 178 236 279 389 |
| | | | 24 | 014 027 035 126 138 157 239 346 789 |
| 30 | 9 | 10 | 25 | 012 034 057 135 146 178 236 279 689 |
| | | | 25 | 014 027 035 123 157 168 269 346 789 |
| 31 | 9 | 10 | 26 | 012 034 078 135 146 179 236 245 389 |
| | | | 27 | 017 024 038 126 139 145 235 346 789 |
| 32 | 9 | 10 | 28 | 012 034 058 069 135 147 189 236 379 |
| | | | 28 | 014 026 035 089 123 158 179 347 369 |
| 33 | 9 | 10 | 29 | 012 034 058 135 147 236 289 469 579 |
| | | | 29 | 014 028 035 123 157 269 346 479 589 |
| 34 | 9 | 11 | 30 | 012 034 067 135 168 245 379 38a 69a |
| | | | 31 | 016 024 037 125 138 345 39a 679 68a |
| 35 | 9 | 11 | 32 | 012 034 135 246 257 39a 489 58a 678 |
| | | | 32 | 013 024 125 267 349 35a 468 578 89a |
| 36 | 10 | 9 | 33 | 012 034 057 068 135 146 178 236 247 258 |
| | | | 34 | 013 026 047 058 124 157 168 235 278 346 |
| 37 | 10 | 9 | 34 | 012 034 057 068 135 146 236 247 258 378 |
| | | | 34 | 015 027 038 046 124 136 235 268 347 578 |

| | Blocks | Points | Config. | |
|---|---|---|---|---|
| 38 | 10 | 9 | 33 | 012 034 057 068 135 146 178 236 247 258 |
| | | | 34 | 013 026 047 058 124 157 168 235 278 346 |
| | | | 34 | 018 027 035 046 125 136 147 234 268 578 |
| 39 | 10 | 9 | 35 | 012 034 058 067 135 147 236 248 257 456 |
| | | | 35 | 013 026 048 057 127 145 234 258 356 467 |
| 40 | 10 | 10 | 36 | 012 034 057 135 146 179 236 258 378 689 |
| | | | 37 | 014 025 037 123 157 169 268 346 358 789 |
| 41 | 10 | 10 | 38 | 012 034 057 089 135 146 179 236 258 459 |
| | | | 39 | 013 028 045 079 126 149 157 235 346 589 |
| 42 | 10 | 10 | 40 | 012 034 057 135 146 189 236 258 279 459 |
| | | | 41 | 013 027 045 126 149 158 235 289 346 579 |
| 43 | 10 | 10 | 42 | 012 034 057 089 135 146 179 236 478 568 |
| | | | 42 | 019 023 047 058 126 134 157 356 468 789 |
| 44 | 10 | 10 | 43 | 012 034 057 089 135 146 236 278 568 679 |
| | | | 43 | 014 028 035 079 123 156 267 346 578 689 |
| 45 | 10 | 10 | 44 | 012 034 056 078 135 146 179 236 245 289 |
| | | | 45 | 017 028 035 046 129 136 145 234 256 789 |
| 46 | 10 | 10 | 45 | 012 034 057 068 135 146 236 245 569 789 |
| | | | 46 | 013 025 046 078 126 145 234 356 579 689 |
| 47 | 10 | 10 | 47 | 012 034 057 068 135 146 236 245 589 679 |
| | | | 47 | 013 024 058 067 126 145 235 346 579 689 |
| 48 | 10 | 10 | 48 | 012 034 057 135 146 178 236 245 279 389 |
| | | | 49 | 017 024 035 126 138 145 239 257 346 789 |
| 49 | 10 | 10 | 50 | 012 034 057 135 146 178 236 245 279 689 |
| | | | 50 | 013 027 045 124 157 168 235 269 346 789 |
| 50 | 10 | 10 | 51 | 012 034 057 135 146 189 236 247 258 459 |
| | | | 51 | 015 023 047 128 136 149 246 257 345 589 |
| 51 | 10 | 10 | 52 | 012 034 057 135 146 189 236 247 258 569 |
| | | | 53 | 015 023 047 128 134 169 246 257 356 589 |
| 52 | 10 | 10 | 54 | 012 034 057 135 146 236 247 258 489 569 |
| | | | 54 | 015 023 047 126 134 248 257 356 469 589 |
| 53 | 10 | 10 | 55 | 012 034 058 069 135 146 178 236 247 289 |
| | | | 56 | 016 024 035 089 123 147 158 269 278 346 |
| 54 | 10 | 10 | 57 | 012 034 058 135 146 178 236 247 289 379 |
| | | | 57 | 014 028 035 127 136 158 239 246 347 789 |
| 55 | 10 | 10 | 58 | 012 034 058 079 135 146 178 236 247 259 |
| | | | 59 | 013 024 059 078 126 147 158 235 279 346 |
| 56 | 10 | 10 | 60 | 012 034 058 135 146 236 247 259 389 679 |
| | | | 60 | 015 024 038 126 134 235 279 369 467 589 |
| 57 | 10 | 10 | 61 | 012 034 135 146 178 236 247 258 389 579 |
| | | | 61 | 013 024 126 147 158 238 257 346 359 789 |
| 58 | 10 | 10 | 62 | 012 034 089 135 146 178 236 247 258 459 |
| | | | 62 | 018 023 049 125 136 147 246 278 345 589 |

| | Blocks | Points | Config. | |
|---|---|---|---|---|
| 59 | 10 | 10 | 63 | 012 034 135 146 178 236 247 258 379 459 |
| | | | 63 | 014 023 126 137 158 245 278 346 359 479 |
| 60 | 10 | 10 | 64 | 012 034 058 067 135 147 189 236 469 579 |
| | | | 64 | 018 026 035 047 123 149 157 346 589 679 |
| 61 | 10 | 10 | 65 | 012 034 058 069 135 147 236 248 257 389 |
| | | | 65 | 014 026 035 089 123 157 247 258 348 369 |
| 62 | 10 | 10 | 66 | 012 034 059 135 147 168 236 248 279 578 |
| | | | 66 | 014 029 035 127 136 158 234 268 478 579 |
| 63 | 10 | 10 | 67 | 012 034 059 135 147 168 236 248 379 567 |
| | | | 67 | 015 024 039 123 148 167 268 347 356 579 |
| 64 | 10 | 10 | 68 | 012 034 068 079 135 147 236 257 289 456 |
| | | | 68 | 017 023 046 089 125 134 268 279 356 457 |
| 65 | 10 | 10 | 69 | 012 034 068 079 135 147 236 257 456 589 |
| | | | 69 | 013 026 047 089 127 145 235 346 568 579 |
| 66 | 10 | 10 | 70 | 012 034 068 135 147 236 257 458 469 789 |
| | | | 70 | 013 026 048 127 145 235 346 479 578 689 |
| 67 | 10 | 10 | 71 | 012 034 089 135 147 236 258 279 459 468 |
| | | | 71 | 013 029 048 127 145 235 268 346 479 589 |
| 68 | 10 | 10 | 72 | 012 034 079 135 147 236 258 489 569 678 |
| | | | 72 | 017 023 049 125 134 268 356 478 589 679 |
| 69 | 10 | 11 | 73 | 012 034 057 089 135 146 178 236 29a 38a |
| | | | 73 | 014 029 035 078 126 138 157 23a 346 89a |
| 70 | 10 | 11 | 74 | 012 034 057 089 135 146 178 236 38a 59a |
| | | | 74 | 014 023 059 078 126 138 157 346 35a 89a |
| 71 | 10 | 11 | 75 | 012 034 057 089 135 146 178 236 29a 68a |
| | | | 76 | 014 029 035 078 123 157 168 26a 346 89a |
| 72 | 10 | 11 | 77 | 012 034 057 135 146 19a 236 278 389 58a |
| | | | 78 | 014 027 035 126 139 15a 238 346 578 89a |
| 73 | 10 | 11 | 79 | 012 034 057 135 146 19a 236 278 589 68a |
| | | | 80 | 014 027 035 123 159 16a 268 346 578 89a |
| 74 | 10 | 11 | 81 | 012 034 057 135 146 236 279 28a 568 59a |
| | | | 81 | 014 027 035 123 156 268 29a 346 579 58a |
| 75 | 10 | 11 | 82 | 012 034 057 135 146 236 279 568 69a 78a |
| | | | 82 | 014 027 035 123 156 269 346 578 68a 79a |
| 76 | 10 | 11 | 83 | 012 034 057 089 135 146 236 245 58a 79a |
| | | | 83 | 013 024 058 079 126 145 235 346 57a 89a |
| 77 | 10 | 11 | 84 | 012 034 078 09a 135 146 179 18a 236 245 |
| | | | 84 | 013 024 079 08a 126 145 178 19a 235 346 |
| 78 | 10 | 11 | 85 | 012 034 078 09a 135 146 236 245 579 58a |
| | | | 85 | 013 024 079 08a 126 145 235 346 578 59a |
| 79 | 10 | 11 | 86 | 012 034 135 146 178 19a 236 245 279 28a |
| | | | 86 | 013 024 126 145 179 18a 235 278 29a 346 |
| 80 | 10 | 11 | 87 | 012 034 135 146 179 18a 236 247 258 29a |
| | | | 87 | 013 024 126 147 158 19a 235 279 28a 346 |

| | Blocks | Points | Config. | |
|---|---|---|---|---|
| 81 | 10 | 11 | 88 | 012 034 135 146 179 236 247 258 59a 78a |
| | | | 88 | 013 024 126 147 159 235 278 346 58a 79a |
| 82 | 10 | 11 | 89 | 012 034 135 146 189 236 247 258 49a 78a |
| | | | 89 | 013 024 126 149 158 235 278 346 47a 89a |
| 83 | 10 | 11 | 90 | 012 034 058 069 135 147 236 27a 389 46a |
| | | | 90 | 015 023 046 089 127 134 26a 358 369 47a |
| 84 | 10 | 11 | 91 | 012 034 058 135 147 236 279 469 48a 56a |
| | | | 91 | 015 023 048 127 134 269 356 46a 479 58a |
| 85 | 10 | 11 | 92 | 012 034 135 147 16a 236 248 259 27a 389 |
| | | | 92 | 014 023 125 136 17a 247 26a 289 348 359 |
| 86 | 10 | 11 | 93 | 012 034 135 147 189 236 278 39a 468 58a |
| | | | 93 | 014 023 127 139 158 268 346 35a 478 89a |
| 87 | 10 | 11 | 94 | 012 034 135 147 189 236 257 38a 456 59a |
| | | | 95 | 014 023 127 138 159 256 346 35a 457 89a |
| 88 | 10 | 11 | 96 | 012 034 135 147 236 258 379 38a 49a 689 |
| | | | 96 | 014 023 125 137 268 34a 358 369 479 89a |
| 89 | 10 | 11 | 97 | 012 034 089 135 147 236 258 27a 459 46a |
| | | | 97 | 013 028 049 127 145 235 26a 346 47a 589 |
| 90 | 10 | 11 | 98 | 012 034 135 147 236 258 379 38a 468 49a |
| | | | 98 | 014 023 125 137 268 346 358 39a 479 48a |
| 91 | 10 | 11 | 99 | 012 034 135 147 236 258 379 468 59a 78a |
| | | | 99 | 014 023 125 137 268 346 359 478 58a 79a |
| 92 | 10 | 11 | 100 | 012 034 135 147 19a 236 258 389 468 78a |
| | | | 100 | 014 023 125 139 17a 268 346 358 478 89a |
| 93 | 10 | 11 | 101 | 012 034 135 147 236 258 378 469 68a 79a |
| | | | 101 | 014 023 125 137 268 346 358 479 69a 78a |
| 94 | 10 | 11 | 102 | 012 034 089 135 147 19a 236 28a 468 567 |
| | | | 102 | 019 023 048 12a 134 157 268 356 467 89a |
| 95 | 10 | 11 | 103 | 012 034 09a 135 147 236 289 38a 468 567 |
| | | | 103 | 014 029 03a 123 157 268 348 356 467 89a |
| 96 | 10 | 11 | 104 | 012 034 135 147 19a 236 289 38a 468 567 |
| | | | 104 | 014 023 129 13a 157 268 348 356 467 89a |
| 97 | 10 | 11 | 105 | 012 034 135 147 19a 236 289 38a 468 578 |
| | | | 105 | 014 023 129 13a 157 268 346 358 478 89a |
| 98 | 10 | 11 | 106 | 012 034 067 135 168 19a 245 279 469 78a |
| | | | 106 | 013 027 046 125 169 18a 249 345 678 79a |
| 99 | 10 | 11 | 107 | 012 034 067 135 168 245 278 29a 469 56a |
| | | | 108 | 013 027 046 128 156 249 25a 345 678 69a |
| 100 | 10 | 11 | 109 | 012 034 067 135 168 245 27a 289 469 56a |
| | | | 109 | 013 027 046 128 156 249 25a 345 67a 689 |
| 101 | 10 | 11 | 110 | 012 034 135 168 19a 246 257 379 489 67a |
| | | | 110 | 013 024 125 16a 189 267 349 357 468 79a |
| 102 | 10 | 11 | 111 | 012 034 09a 135 18a 246 257 368 379 67a |
| | | | 111 | 01a 024 039 125 138 267 346 357 68a 79a |

| | Blocks | Points | Config. | |
|---|---|---|---|---|
| 103 | 10 | 12 | 112 | 012 034 135 146 178 236 379 58a 59b 7ab |
| | | | 112 | 014 023 126 137 158 346 359 5ab 78a 79b |
| 104 | 10 | 12 | 113 | 012 034 135 146 178 236 379 3ab 58a 89b |
| | | | 113 | 014 023 126 137 158 346 35a 39b 789 8ab |
| 105 | 10 | 12 | 114 | 012 034 078 09a 135 146 236 245 79b 8ab |
| | | | 114 | 013 024 079 08a 126 145 235 346 78b 9ab |
| 106 | 10 | 12 | 115 | 012 034 135 146 178 19a 236 245 79b 8ab |
| | | | 115 | 013 024 126 145 179 18a 235 346 78b 9ab |
| 107 | 10 | 12 | 116 | 012 034 135 147 168 19a 236 378 39b 5ab |
| | | | 116 | 014 023 126 139 15a 178 347 35b 368 9ab |
| 108 | 10 | 12 | 117 | 012 034 135 147 19a 236 278 39b 468 5ab |
| | | | 117 | 014 023 127 139 15a 268 346 35b 478 9ab |
| 109 | 10 | 12 | 118 | 012 034 067 09a 135 168 19b 245 378 6ab |
| | | | 118 | 019 024 037 06a 125 138 16b 345 678 9ab |
| 110 | 10 | 12 | 119 | 012 034 067 09a 135 168 245 29b 378 4ab |
| | | | 120 | 016 029 037 04a 125 138 24b 345 678 9ab |
| 111 | 10 | 12 | 121 | 012 034 067 135 168 245 378 69a 79b 8ab |
| | | | 121 | 016 024 037 125 138 345 679 68a 78b 9ab |
| 112 | 10 | 12 | 122 | 012 034 067 09a 135 168 19b 245 6ab 789 |
| | | | 122 | 013 024 06a 079 125 16b 189 345 678 9ab |
| 113 | 10 | 12 | 123 | 012 034 067 08a 09b 135 245 568 579 5ab |
| | | | 123 | 013 024 068 079 0ab 125 345 567 58a 59b |
| 114 | 10 | 12 | 124 | 012 034 067 08a 135 245 568 579 7ab 89b |
| | | | 124 | 013 024 068 07a 125 345 567 589 79b 8ab |
| 115 | 10 | 12 | 125 | 012 034 067 09a 135 245 568 59b 6ab 789 |
| | | | 125 | 013 024 06a 079 125 345 56b 589 678 9ab |
| 116 | 10 | 12 | 126 | 012 034 135 246 257 28a 29b 368 379 3ab |
| | | | 126 | 013 024 125 268 279 2ab 346 357 38a 39b |
| 117 | 10 | 12 | 127 | 012 034 135 246 257 28a 368 379 7ab 89b |
| | | | 127 | 013 024 125 268 27a 346 357 389 79b 8ab |
| 118 | 10 | 12 | 128 | 012 034 135 246 257 368 58a 679 7ab 89b |
| | | | 128 | 013 024 125 267 346 358 57a 689 79b 8ab |
| 119 | 10 | 12 | 129 | 012 034 135 246 257 368 3ab 59a 679 89b |
| | | | 129 | 013 024 125 267 346 35a 38b 579 689 9ab |
| 120 | 10 | 12 | 130 | 012 034 135 246 257 39a 489 5ab 678 79b |
| | | | 130 | 013 024 125 267 349 35a 468 57b 789 9ab |
| 121 | 10 | 13 | 131 | 012 034 135 146 236 245 789 7ab 8ac 9bc |
| | | | 131 | 013 024 126 145 235 346 78a 79b 89c abc |
| 122 | 10 | 13 | 132 | 012 034 067 089 135 245 68a 79b 7ac 8bc |
| | | | 132 | 013 024 068 079 125 345 67a 7bc 89b 8ac |
| 123 | 10 | 13 | 133 | 012 034 067 089 0bc 135 245 68a 7ab 9ac |
| | | | 133 | 013 024 068 07b 09c 125 345 67a 89a abc |
| 124 | 10 | 14 | 134 | 012 034 135 245 678 69a 79b 8ac 8bd 9cd |
| | | | 134 | 013 024 125 345 679 68a 78b 8cd 9ac 9bd |

# Chapter 6

# Distance and fractional isomorphism

In this chapter we confine our attention mainly to the 80 pairwise non-isomorphic STS(15)s, and we refer to them by the standard numbering as given in [16, Chapter 5]. The results appear in [23].

Recall that a *trade* $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ is a pair of disjoint $m$-block configurations $\mathcal{T}_1$ and $\mathcal{T}_2$ which has the property that every pair of distinct elements occurs in precisely the same number (zero or one) of triples of $\mathcal{T}_1$ as of $\mathcal{T}_2$. It is well-known that there exist trades of volume $n$ only for $n = 4$ and $n \geq 6$ [46]; for example, there is a unique trade of volume 4, called a *Pasch switch.* A complete list of trades of up to 9 blocks is given in [22] and up to 10 blocks in Chapter 5 of this thesis, from either of which it can be seen that every trade, $\{\mathcal{T}_1, \mathcal{T}_2\}$, of volume not exceeding 8 has $\mathcal{T}_1 \cong \mathcal{T}_2$. If $S = (V, \mathcal{B})$ and $S' = (V, \mathcal{B}')$ are two Steiner triple systems and $T = \{\mathcal{C}, \mathcal{D}\}$ is a pair of configurations with $\mathcal{C} \subseteq \mathcal{B}$ such that $S'$ is isomorphic to $(V, (\mathcal{B} \setminus \mathcal{C}) \cup \mathcal{D})$, then we say that $T$ *transforms* $S$ *into* $S'$.

We define the *distance*, $d(S, S')$, between Steiner triple systems $S$ and $S'$ to be the smallest possible number of blocks of a configuration $\mathcal{C}$ which forms part of a trade $\{\mathcal{C}, \mathcal{D}\}$ that transforms $S$ to $S'$. We investigate the distance problem for STS(15)s and we note in passing that the distance between the cyclic STS(13) and the non-cyclic STS(13) is 4.

Motivated by a paper [58] of Quattrocchi and Rinaldi, who introduce the concept of fractional isomorphism, we say that two configurations $\mathcal{C}$ and $\mathcal{D}$ are $n^{-1}$-*isomorphic* if there are partitions $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$ of $\mathcal{C}$ and $(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)$ of $\mathcal{D}$

such that $\mathcal{C}_i \cong \mathcal{D}_i$ for $i = 1, 2, \ldots, n$. Two Steiner triple systems, $(V, \mathcal{B})$ and $(V', \mathcal{B}')$, are $n^{-1}$-isomorphic if $\mathcal{B}$ and $\mathcal{B}'$ are $n^{-1}$-isomorphic. For $n \geq 2$, two configurations are said to be *strictly $n^{-1}$-isomorphic* if they are $n^{-1}$-isomorphic but not $(n-1)^{-1}$-isomorphic; similarly for Steiner triple systems. Clearly $1^{-1}$-isomorphism is the same as isomorphism. However, unlike isomorphism, $n^{-1}$-isomorphism is not necessarily an equivalence relation if $n \geq 2$; reflexivity and symmetry are always satisfied but in general transitivity fails.

If the trade $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ transforms $S$ to $S'$ and $\mathcal{T}_1 \cong \mathcal{T}_2$, then $S$ and $S'$ are $2^{-1}$-isomorphic. However, as noted in [22], there are trades consisting of non-isomorphic tradeable configurations. We ask the following question. For two non-isomorphic STS($v$)s, $S$ and $S'$, what is the minimum volume of a trade $T$ consisting of isomorphic tradeable configurations, which transforms $S$ into (a system isomorphic to) $S'$? Formally, we define this to be $h(S, S')$. If no such trade exists, we write $h(S, S') = \infty$. If $h(S, S') < \infty$, then $S$ and $S'$ are $2^{-1}$-isomorphic. Although exceptions are relatively scarce, the converse is not necessarily true, as our investigations of the $h$ function for STS(15)s will reveal.

## 6.1 Fractional isomorphism

Kirkman [45] gave the first proof that for every admissible $v$ there exists an STS($v$). Later, Moore [56] proved that for all admissible $v \geq 13$, there exist two non-isomorphic STS($v$)s; see [16, page 70]. We now state and prove two existence theorems concerning $n^{-1}$-isomorphic STS($v$)s.

**Theorem 6.1.1** *For all admissible $v \geq 13$, there exist two STS($v$)s which are strictly $2^{-1}$-isomorphic.*

We conjecture that for each positive integer $n$, there exists $v_0(n)$ such that for all admissible $v \geq v_0(n)$ there exist two STS($v$)s which are strictly $n^{-1}$-isomorphic. Whilst we are unable to prove this conjecture, we can establish a weaker result in the same direction.

**Theorem 6.1.2** *Let $N$ be a given positive integer. Then for all sufficiently large admissible $v$, there exists STS($v$)s, $S$ and $S'$ such that for some $n > N$, $S$ is strictly*

*$n^{-1}$-isomorphic to $S'$.*

Before dealing with these theorems we prove some lemmas.

**Lemma 6.1.1** *If $\mathcal{X}$ is a configuration, let $\rho(\mathcal{X})$ denote the (possibly empty) set of blocks obtained by removing from $\mathcal{X}$ all blocks containing points of degree 1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are configurations which cover the same pairs, and suppose also that $\mathcal{C} \cong \mathcal{D}$. Then $\rho(\mathcal{C}) \cong \rho(\mathcal{D})$ and $\rho(\mathcal{C})$ covers the same pairs as $\rho(\mathcal{D})$.*

**Proof.** If $\mathcal{C}$ contains no points of degree 1, there is nothing to prove.

Otherwise let $\tau : P(\mathcal{C}) \to P(\mathcal{D})$ be an isomorphism from $\mathcal{C}$ to $\mathcal{D}$. Let $\mathcal{A}$ be the set of blocks of $\mathcal{C}$ which contain points of degree 1. Since $\mathcal{C}$ and $\mathcal{D}$ cover the same pairs, a block containing a point of degree 1 in one of the configurations $\mathcal{C}$ and $\mathcal{D}$ must also occur in the other configuration. Therefore $\mathcal{A} \subseteq \mathcal{C} \cap \mathcal{D}$. Then $\rho(\mathcal{C}) = \mathcal{C} \setminus \mathcal{A}$ and $\rho(\mathcal{D}) = \mathcal{D} \setminus \mathcal{A}$. Since $\tau(\mathcal{A}) = \mathcal{A}$, we have $\rho(\mathcal{C}) \cong \rho(\mathcal{D})$. Furthermore, since we have removed the same pairs from $\mathcal{C}$ and $\mathcal{D}$, the configurations $\rho(\mathcal{C})$ and $\rho(\mathcal{D})$ cover the same pairs. □

**Lemma 6.1.2** *Suppose $S$ and $S'$ are Steiner triple systems and that $n \geq 1$. If there exists a trade $\{\mathcal{C}, \mathcal{D}\}$ with $\mathcal{C}$ $n^{-1}$-isomorphic to $\mathcal{D}$ that transforms $S$ to $S'$, then $S$ is $(n+1)^{-1}$-isomorphic to $S'$.*

**Proof.** This follows directly from the definitions. □

The converse of Lemma 6.1.2 is not true. In an attempt to identify the reason for this, we define a *pseudo-trade* as a pair of configurations $U = \{\mathcal{C}, \mathcal{D}\}$ such that $\mathcal{C}$ and $\mathcal{D}$ cover the same pairs, $\mathcal{C} \cong \mathcal{D}$, $\mathcal{C} \cap \mathcal{D} \neq \emptyset$, and for any non-empty subset $\mathcal{A}$ of $\mathcal{C} \cap \mathcal{D}$ we have $\mathcal{C} \setminus \mathcal{A} \ncong \mathcal{D} \setminus \mathcal{A}$. By Lemma 6.1.1, $\mathcal{C}$ and $\mathcal{D}$ have no points of degree 1.

Pseudo-trades of small volume may be enumerated by methods similar to those described in [22] and Chapter 5 of this thesis. An example of a pseudo-trade is given by $\{\mathcal{C}, \mathcal{D}\} = \{\{012, 034, 056, 135, 146, 179, 1bc, 245, 37b, 47c, 49b\}, \{016, 024, 035, 125, 13b, 14c, 179, 347, 456, 49b, 7bc\}\}$, where $\mathcal{C} \cap \mathcal{D} = \{179, 49b\}$. The table below gives, for small volumes, the number of labelled pseudo-trades $\{\mathcal{C}, \mathcal{D}\}$ where configuration $\mathcal{C}$ is canonically labelled (see section 5.2).

| $\lvert\mathcal{C}\rvert$ | $\leq 10$ | 11 | 12 | 13 |
|---|---|---|---|---|
| pseudo-trades $\{\mathcal{C},\mathcal{D}\}$ | 0 | 8 | 24 | 168 |

With the definition of pseudo-trades in place we have the following result.

**Lemma 6.1.3** *Let $S = (V, \mathcal{B})$ and $S' = (V, \mathcal{B}')$ be strictly $2^{-1}$-isomorphic Steiner triple systems. Let $\beta = (|\mathcal{B}| - 1)/2$. Then there exist $T = \{\mathcal{C}, \mathcal{D}\}$ with $\mathcal{C} \cong \mathcal{D}$ and $|\mathcal{C}| \leq \beta$ where $T$ is either a trade or a pseudo-trade and $T$ transforms $S$ to $S'$.*

**Proof.** Suppose there exists a $2^{-1}$-isomorphism consisting of a partition of $\mathcal{B}$ into $\mathcal{B}_0$ and $\mathcal{B}_1$ with $|\mathcal{B}_1| \leq \beta$, a partition of $\mathcal{B}'$ into $\mathcal{B}'_0$ and $\mathcal{B}'_1$ with $|\mathcal{B}'_1| \leq \beta$, and one-to-one mappings $\phi_0 : V \to V$ and $\phi_1 : V \to V$ such that $\phi_0(\mathcal{B}_0) = \mathcal{B}'_0$ and $\phi_1(\mathcal{B}_1) = \mathcal{B}'_1$. Apply $\phi_0^{-1}$ to $S'$, let $\mathcal{B}''_1 = \phi_0^{-1}(\mathcal{B}'_1)$ and consider the pair $\{\mathcal{B}_1, \mathcal{B}''_1\}$. Note that $|\mathcal{B}_1| \leq \beta$ and that $(\mathcal{B} \setminus \mathcal{B}_1) \cup \mathcal{B}''_1 \cong \mathcal{B}'$.

Let $\mathcal{F} = \mathcal{B}_1 \cap \mathcal{B}''_1$. If $\mathcal{F} = \emptyset$ then $\{\mathcal{B}_1, \mathcal{B}''_1\}$ is a trade which satisfies the conditions of the lemma. So we may assume that $\mathcal{F}$ is non-empty. If $\{\mathcal{B}_1, \mathcal{B}''_1\}$ is a pseudo-trade, we are done. Otherwise there exists a non-empty set $\mathcal{G}$ of maximum cardinality such that $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{B}_1 \setminus \mathcal{G} \cong \mathcal{B}''_1 \setminus \mathcal{G}$. It is clear from the definition that $\{\mathcal{B}_1 \setminus \mathcal{G}, \mathcal{B}''_1 \setminus \mathcal{G}\}$ is a trade or a pseudo-trade with the required properties. □

For a result which is surely obvious, the proof of the next lemma is more complex than we would have desired. We use it to prove Theorem 6.1.1.

**Lemma 6.1.4** *For all admissible $v \geq 27$, there exists an $STS(v)$ which contains precisely one sub-STS(13).*

**Proof.** For admissible $v$ such that $27 \leq v \leq 63$, it is straightforward to generate STS($v$)s with the desired property by Stinson's hill-climbing method [68].

For admissible $v > 63$ we employ a recursive construction. Let $G$ be a $\{3\}$-GDD of type $g^t h^u$ and suppose we have an STS($g+13$) and an STS($h+13$), each having a unique sub-STS(13). Construct a new Steiner triple system, $S$, of order $tg+uh+13$ as follows. Let $T$ be an STS(13). On each group of size $g$, together with the points of $T$, put an STS($g+13$) such that the sub-STS(13) coincides with $T$. Similarly, on each group of size $h$, together with the points of $T$, put an STS($h+13$) such that the sub-STS(13) coincides with $T$.

Suppose, further, that $G$ has at most four groups; i.e. $t + u \leq 4$. We show that the system $S$ has a unique STS(13), namely $T$. To prove this, suppose $U$ is a sub-STS(13) of $S$ and $U \neq T$. Label the groups $G_1, G_2, \ldots, G_n$, where $n = 3$ or 4. Let $A_i$ be the set of points of $U$ which lie on $G_i$ and let $A$ be the set of points which are common to both $U$ and $T$. Consider three cases according to the size of $A$.

(i) $|A| = 0$. For each $i$, we must have $|A_i| = 0$, 1 or 3. (We can rule out $|A_i| = 7$ and $|A_i| = 9$ because we know that neither STS(13) has a sub-STS(7) or a sub-STS(9).) As there are at most four groups, $U$ cannot exist.

(ii) $|A| = 1$. Similarly we must have $|A_i| = 0$ or 2. Again, there are insufficient groups for $U$ to exist.

(iii) $|A| = 3$. Now we are forced to have $|A_i| = 0$ for all $i$.

Thus the construction described above preserves the property of containing a unique sub-STS(13). By a theorem of Colbourn, Hoffman and Rees [13], there exist $\{3\}$-GDDs of the following types:

$$\begin{array}{l} g^3 \\ g^3h^1, \quad g \equiv h \equiv 0 \ (\mathrm{mod}\ 2),\ h \leq 2g \end{array}$$

Using the construction with $\{3\}$-GDDs of these types and starter systems of orders 27, 31, $\ldots$, 63, we can generate the desired STS($v$)s for all admissible $v > 63$ as follows.

First we construct a suitable STS(67) using a $\{3\}$-GDD of type $18^3$. Then, using $\{3\}$-GDDs of type $g^3h^1$ with $g = 14$ and $h = 14$, 18, 20, 24 and 26, we construct suitable STS($v$)s for $v = 69$, 73, 75, 79 and 81, respectively. Now let $k \geq 4$ and suppose that we already have suitable STS($u$)s for admissible $u$ in the range $27 \leq u \leq 3^k$. Let an admissible $v$ be given such that $3^k < v \leq 3^{k+1}$, and write $v = 6r + e$, where $e = 1$ or 3. If $r \equiv 0$ or 1 (mod 3), put $s = 2r + 1$ and $t = 36 + e$; otherwise put $s = 2r + 3$ and $t = 30 + e$. Let $g = s - 13$ and $h = t - 13$. Then in either case $2g - h \geq 4r - e - 47 \geq 0$, since $r \geq 14$ and $e \leq 3$. It is easily verified that $27 \leq s, t \leq 3^k$ for admissible $s$ and $t$; hence we can use a $\{3\}$-GDD of type $g^3h^1$ to construct a suitable Steiner triple system of order $3g+h+13 = v$. $\square$

**Proof of Theorem 6.1.1.** The two STS(13)s are $2^{-1}$-isomorphic because one can be transformed into the other by a Pasch trade. For the same reason, STS(15) #1 is $2^{-1}$-isomorphic to STS(15) #2. Pairs of $2^{-1}$-isomorphic STS($v$)s for $v = 19$, 21 and 25 are easily produced by choosing an appropriate system and transforming it by a Pasch trade.

So let $v \geq 27$ and let $S$ be an STS($v$) which contains a unique sub-STS(13), $T$, say. Lemma 6.1.4 guarantees that $S$ exists. Choose a Pasch configuration in $T$ which when traded transforms $T$ into an STS(13) of the other isomorphism type. Perform this trade thus transforming $S$ into $S'$, say. By Lemma 6.1.2, $S$ is $2^{-1}$-isomorphic to $S'$, but clearly $S$ is not isomorphic to $S'$. □

**Proof of Theorem 6.1.2.** On the one hand, the number of distinct STS($v$)s is $v^{v^2(1/6+o(1))}$ as $v \to \infty$ [70]. On the other hand, consider an STS($v$), say $S$, with sufficiently large $v$. Partition the $v(v-1)/6$ blocks of $S$ into $N$ not necessarily non-empty sets. Such a partition can be represented by a vector of length $v(v-1)/6$ with entries from 1 to $N$. Therefore the number of possible partitions is at most $N^{v(v-1)/6}$. For each set of the partition, we can apply a permutation to the base set. Since the number of combinations of permutations is $(v!)^N$, this process gives at most $(v!)^N N^{v(v-1)/6}$ STS($v$)s which are $N^{-1}$-isomorphic to $S$. Any such systems (on the same base set) must arise at least once in this manner. □

## 6.2 Algorithms

The other main results of this chapter are two matrices, $D = [d_{i,j}]$ and $H = [h_{i,j}]$, showing relations between Steiner triple systems of order 15. The first is the 'distance table' for STS(15)s, where $d_{i,j}$ is the volume of the smallest trade that transforms STS(15) #$i$ into STS(15) #$j$, the numbers $i$ and $j$ referring to the standard numbering of the 80 STS(15)s. In the second matrix, $H$, the entry $h_{i,j}$ is the volume of the smallest trade between isomorphic configurations which transforms STS(15) #$i$ into STS(15) #$j$. We describe two algorithms for computing $[d_{i,j}]$ and $[h_{i,j}]$.

**Algorithm 6.2.1**

For $b = 4, 6, 7, 8, \ldots$, make a list, $L_b$, of all possible trades and pseudo-trades $\{\mathcal{C}, \mathcal{D}\}$, where $\mathcal{C}$ is a $b$-block configuration which can occur in an STS(15).

For each STS(15), $S$, for each $\{\mathcal{C}, \mathcal{D}\} \in L_b$:

For each occurrence $\phi(\mathcal{C})$ of an isomorphic copy of $\mathcal{C}$ in $S$: transform $S$ to $S'$, say, by the trade or pseudo-trade $\{\phi(\mathcal{C}), \phi(\mathcal{D})\}$. Record the designation (#01 – #80) of $S$ and $S'$ as well as information about the trade.

**Algorithm 6.2.2**

For $b = 4, 6, 7, 8, \ldots$, for each STS(15), $S$, for each set $\mathcal{C}$ of $b$ blocks of $S$:

For each trade or pseudo-trade $\{\mathcal{C}, \mathcal{D}\}$: record the designation of $S$ and $S'$, the STS(15) that results from transforming $S$ by $\{\mathcal{C}, \mathcal{D}\}$, as well as information about $\{\mathcal{C}, \mathcal{D}\}$.

In spite of its apparent naivety, Algorithm 6.2.2 is the preferred option. It turns out that Algorithm 6.2.1 is not practicable for dealing with $b \geq 10$ because of the difficulty of constructing the list $L_b$. On the other hand, Algorithm 6.2.2 does not require a predetermined list and, furthermore, there is an efficient method, namely Algorithm 5.2.1, for constructing all possible trades $\{\mathcal{C}, \mathcal{D}\}$, if any, from a given configuration $\mathcal{C}$. Also it is clear how to adapt the Algorithm 5.2.1 to construct pseudo-trades. In fact, we used both methods for $b \leq 9$ and thereby gave ourselves extra confidence that our computer programming was sound.

There are a number of ways to shorten the computational effort and reduce the amount of work to a reasonable level. We mention three observations. (i) A configuration that is part of a trade or a pseudo-trade has no points of degree one. (ii) To prove that two STS(15)s are $2^{-1}$-isomorphic, we do not need to consider trades or pseudo-trades of volume greater than 17. This follows from the proof of Lemma 6.1.3. (iii) In computing the matrix $H$, after examining all trades of volume less than or equal to 17, a complete list of pairs $(i, j)$ where $h_{i,j} > 17$ is known. If in addition we know that the smallest pseudo-trade which transforms STS(15) $\#i$ to STS(15) $\#j$ has volume $p \leq 17$, we can deduce that either $h_{i,j} \leq 35 - p$ or $h_{i,j} = \infty$, thus further limiting the search space.

## 6.3 Results

The two matrices $D$ and $H$ are presented in tabular form. For clarity, only the upper half of the matrix is given; the other half follows by symmetry.

In Table 6.3.1 (page 89), the entry $(i, j)$, $i \leq j$, indicates $d_{i,j}$, the volume of the smallest trade that transforms STS(15) #$i$ to STS(15) #$j$. We do not distinguish between trades with isomorphic configurations and trades with non-isomorphic configurations. Numbers 10, 11, ..., 19 are represented by lower-case letters a, b, ..., j, respectively. We find that any STS(15) can be transformed into any other STS(15) by a trade of at most 19 blocks. Also 19 blocks are necessary only for the pairs {STS(15) #01, STS(15) #62} and {STS(15) #01, STS(15) #71}. Eighteen blocks suffice for the rest. If STS(15) #01 is excluded, then 17 blocks are sufficient and sometimes necessary.

Table 6.3.2 (page 91) has the same format as Table 6.3.1 except that each trade consists of a pair of isomorphic configurations. The entry $(i, j)$, $i \leq j$, indicates $h_{i,j}$, the volume of the smallest such trade that transforms STS(15) #$i$ to STS(15) #$j$. A dot indicates that no such trade exists: $h_{i,j} = \infty$. The same scheme as above is used for representing two-digit numbers, and entries that differ from the corresponding values in Table 6.3.1 are underlined. (Observe that values 4, 6, 7 and 8 occur at precisely the same locations in both tables.) There are only two values greater than 17: $h_{06,31} = 20$, represented by the letter k in the table, and $h_{07,25} = 24$, represented by the letter o.

Let $\nu$ be the smallest $n$ such that any two STS(15)s are $n^{-1}$-isomorphic. Lemma 6.1.3 and the existence of pairs $(i, j)$ where $h_{i,j} = \infty$ and $d_{i,j} > 17$ (at $(01, 71)$ for example) imply that $\nu \geq 3$. However, from the information in Table 6.3.2 it is easy to deduce that $\nu \leq 4$. The table shows that STS(15) #11 is $2^{-1}$-isomorphic to every other STS(15) except possibly STS(15) #01. Therefore it follows from Proposition 6 of [58] that for $2 \leq i < j \leq 80$, STS(15) #$i$ is $4^{-1}$-isomorphic to STS(15) #$j$. In a similar manner we can show that STS(15) #01 is $4^{-1}$-isomorphic to STS(15) #$j$ for $2 \leq j \leq 80$ by identifying a system STS(15) #$k$ which is $2^{-1}$-isomorphic to both STS(15) #01 and STS(15) #$j$. With a lot more work we will show that $\nu = 3$.

Table 6.3.2 shows that all except 537 pairs of STS(15)s are $2^{-1}$-isomorphic.

However, to ascertain the full extent of $2^{-1}$-isomorphism we must also determine $2^{-1}$-isomorphic pairs of STS(15)s which are not indicated by Table 6.3.2. It suffices, by Lemma 6.1.3, to look for pseudo-trades of volume not greater than 17. A complete search produces a further six $2^{-1}$-isomorphic pairs. Specifically, let $e_{i,j}$ denote the smallest volume of a pseudo-trade, if any, that transforms STS(15) $\#i$ to STS(15) $\#j$. Then we have the following values for pairs $(i, j)$ where $i < j$ and $h_{i,j} = \infty$.

| $i, j$ | $05, 30$ | $05, 34$ | $12, 71$ | $16, 29$ | $19, 67$ | $19, 72$ |
|---|---|---|---|---|---|---|
| $e_{i,j}$ | 15 | 15 | 15 | 15 | 17 | 16 |

It is also worth mentioning that in the only two cases where $17 < h_{i,j} < \infty$ we have $e_{06,31} = e_{07,25} = 11$.

Thus $2^{-1}$-isomorphism accounts for all except $537 - 6 = 531$ pairs of STS(15)s. To establish a $3^{-1}$-isomorphism for the remaining pairs, three approaches may be used. Let $S$ and $S'$ be STS(15)s which are not $2^{-1}$-isomorphic. (i) As in the proof of Proposition 6 of [58], it is sufficient to find an STS(15), $S''$, and trades, $\{\mathcal{C}, \mathcal{D}\}$ and $\{\mathcal{E}, \mathcal{F}\}$, where $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{E} \cong \mathcal{F}$, such that $\{\mathcal{C}, \mathcal{D}\}$ transforms $S''$ to $S$, $\{\mathcal{E}, \mathcal{F}\}$ transforms $S''$ to $S'$ and either $\mathcal{C} \bigcap \mathcal{E} = \emptyset$, or $\mathcal{C} \subseteq \mathcal{E}$, or $\mathcal{E} \subseteq \mathcal{C}$. (ii) We find a trade that consists of $2^{-1}$-isomorphic configurations, possibly the one which was used to establish the value of the corresponding entry in Table 6.3.1, and then apply Lemma 6.1.2. (iii) We find a trade, $\{\mathcal{C}, \mathcal{D}\}$ that transforms $S$ to $S'$ and a set of blocks $\mathcal{X}$ of $S$ disjoint from $\mathcal{C}$ such that $\mathcal{C} \cup \mathcal{X}$ is $2^{-1}$-isomorphic to $\mathcal{D} \cup \mathcal{X}$.

Elementary computation shows that every trade of volume at most 12 consists of a pair of $2^{-1}$-isomorphic configurations. This accounts for every pair where there is a value of c or less in Table 6.3.1. With a little more computation we can use the same method to establish the required $2^{-1}$-isomorphism for the trades corresponding to entries in Table 6.3.1 with values d, e, f and g. Hence for pairs $\{i, j\}$ where there is one of these letters in Table 6.3.1 and a dot in Table 6.3.2, we have that STS(15) $\#i$ is $3^{-1}$-isomorphic to STS(15) $\#j$.

Of the remaining 39 cases, where the value in Table 6.3.1 is h, i or j, $3^{-1}$-isomorphic pairs have been found; four by method (i): $(01, j)$, $j = 33, 64, 76, 79$; a further 33 pairs by method (ii): $(01, j)$, $j =$ 36, 37, 38, 41, 44, 45, 46, 48, 49, 50, 52, 53, 55, 56, 57, 58, 60, 61, 63, 65, 66, 67, 68, 69, 70, 71, 72, 74, 75, 77, as well as

(02, 77), (03, 80) and (16, 80); and two pairs by method (iii): (01, 43) and (01, 62). This completes the proof of the following.

**Theorem 6.3.1** *Any two Steiner triple systems of order 15 are* $3^{-1}$*-isomorphic.*

Two particular cases of the final 39, namely (01, 43) and (01, 62), required considerable amounts of computer time, mainly because methods (i) and (ii) failed to produce the desired results. So it is appropriate to give the details of these $3^{-1}$-isomorphisms. In the first case we have:

STS(15) #43: 012 034 057 06a 08c 09d 0be 135 146 17c 189 1ae 1bd 236 247 258 29a 2bc 2de 37d 38b 39e 3ac 459 48e 4ab 4cd 56b 5ad 5ce 67e 68d 69c 78a 79b;

extended trade: (06a 08c 09d 135 17c 189 1ae 247 29a 39e 3ac 459 4cd 5ad 5ce 68d 69c 78a 146, 069 08a 0cd 139 15a 178 1ce 249 27a 35c 3ae 45d 47c 59e 68c 6ad 89d 9ac 146) $= (\mathcal{C}, \mathcal{D})$;

$\mathcal{C}_1$: 08c 09d 29a 39e 4cd 5ad 68d 69c 78a,

$\mathcal{D}_1$: 069 08a 139 15a 178 249 68c 6ad 89d,

$\mathcal{C}_2$: 06a 135 17c 189 1ae 247 3ac 459 5ce 146,

$\mathcal{D}_2$: 0cd 1ce 27a 35c 3ae 45d 47c 59e 9ac 146.

And in the second case:

STS(15) #62: 012 034 057 068 09b 0ad 0ce 135 146 17a 18b 19e 1cd 236 245 27b 28c 29d 2ae 37c 38d 39a 3be 47d 48e 49c 4ab 569 58a 5bc 5de 67e 6ac 6bd 789;

extended trade: (012 034 09b 0ad 0ce 17a 19e 1cd 29d 37c 39a 3be 49c 4ab 5bc 5de 67e 6ac 6bd 245, 01e 02d 03c 04b 09a 129 17c 1ad 349 37e 3ab 4ac 5bd 5ce 67a 6be 6cd 9bc 9de 245) $= (\mathcal{C}, \mathcal{D})$;

$\mathcal{C}_1$: 09b 19e 1cd 29d 3be 49c 5bc 67e 6ac 6bd,

$\mathcal{D}_1$: 01e 02d 03c 09a 1ad 4ac 5ce 67a 6be 6cd,

$\mathcal{C}_2$: 012 034 0ad 0ce 17a 37c 39a 4ab 5de 245,

$\mathcal{D}_2$: 04b 129 17c 349 37e 3ab 5bd 9bc 9de 245.

In each case one can verify that the extended trade transforms the given STS(15) into STS(15) #01.

Table 6.3.1: $d_{i,j}$, the distance between STS(15)s, part I

| | 0000000001111111111222222222233333333334<br>1234567890123456789012345678901234567890 | 4444444445555555555666666666677777777778<br>1234567890123456789012345678901234567890 |
|---|---:|---|
| 01 | 04688ccaccdccae7deefffgeedffggghggiihgg | hgiiiigiiighhghhihghhjiiiihiiijhgiiiigif |
| 02 | 044488688a886a6aacccccccacccdcdedefgfdd | eeggffefffeeedeefedfefefffefffgfeffehegf |
| 03 | 04486487a7648478cbbbba98aabcbcdddefddc | ceeeeedeeecdccddedddefeeefeefefeeefegefh |
| 04 | 044844474446886a8998886998a8acbccedba | bcedcdccddcccbcccbadcccdcdcddcddecdcecef |
| 05 | 04446477468868a9bb8988aaab8bcccdedcc | ccddcdcdddcdcccddcbdcddddecddcedccddfcef |
| 06 | 0486647774c8488999999889898aabcccbb | bcccccbcccccbbcccbccacacccdccccddccddddf |
| 07 | 08977b678a48abbcac9abbbaaaccdecdcd | dcccecceddcfeddedddccfdddededdeecedfdccf |
| 08 | 0448744444488bb776477797aa9abdba9 | 9cbbbcbbcbaba9abbbabcbacccccbcccbcaecdf |
| 09 | 044744487484994446664777889 9cb99 | 9bba9b9ababa9a9aa97a99aaabbbb9bcdaa8ccde |
| 10 | 04444487477994644777747989acaa9 | 9baa9b9bbbaba99ab99aaaababbabbb9b9dabe |
| 11 | 04774c774466667744747477 98b988 | 8a889b89999a98aa988a897aa9aaa8aab989aaac |
| 12 | 0777ab86466444488476788aadb88 | 8bba9b9abbaa999bb87a9a9aababbabbdaa8dcde |
| 13 | 04484477bb7747886879989beab9 | 9aaabbabbbabaa9cbaaabbacbccbccccbbbdddd |
| 14 | 07667b7bb7764888989a9aaeba9 | 9cccbcbbcbaca9babaabbcccccccadcecdbdcef |
| 15 | 08444799777747787 7aab8aaab | 9aaabcaab9baaaab99aaba7ababbaabbb9aadcce |
| 16 | 068ccddaa97bbbcbcdbcdfddb | cedededeeecdcbddeeddfedeeefeeefefefdgffh |
| 17 | 0477bb9a77779aaa99bacbbb | baaabacbbbaccbbcacaaccacccbdbacccbcddccd |
| 18 | 074a97777747a7977bbb99a | bcba9ba9ab9aa9aaab8a9a9ababbb8abcabbdaef |
| 19 | 0668999977a79799aacb99 | 9baaab8bb9bbaaccbaaa9a4acbbbbabbababcaac |
| 20 | 086777777779744aac9aa | abba99a999989899a84869989bb998aabaaaabce |
| 21 | 0498889888a8aa499844 | 4a998848889a77a9a7ab67899aa9a99aa8a9bbbd |
| 22 | 0899989a9a8888bb977 | 79baaa7a99884488848944a79999aa9ababbbadd |
| 23 | 04444477447497b988 | 88a76868899988a89687b87a8888978ab877aabd |
| 24 | 04444764444a7a968 | 8887886888976898987 7b8889888979ab874a9bc |
| 25 | 044444766468b886 | 6a779889888888698877b9799a9a98a9ba97baad |
| 26 | 04447477787b986 | 6a998a88a88986888889b9899aaa97a9b997babe |
| 27 | 0474747794b888 | 68779a88869888896679a948998887988677aaac |
| 28 | 0777744878668 | 8aa96886886786889947a9778998a478a897b8cc |
| 29 | 0874478aca88 | 6ba788999897899a8969a668a9a9a8abbb7aaacd |
| 30 | 0444684b886 | 68448888999868889879aa77888a8798b879977d |
| 31 | 0777a8a899 | 9b778b8baaba9999a9a9caaaa8bab99a8697c88c |
| 32 | 06487a968 | 68848868688766889847a64886899478884877ac |
| 33 | 0487a888 | 887866886674768786778994888884888897a99b |
| 34 | 089a688 | 889786868867668686749977688864888888a8bc |
| 35 | 08b474 | 4b86787798774898a899a6669a977899a7bb98bd |
| 36 | 08896 | 8a44888989879a9999abcb7a98b8a99bc8aab88c |
| 37 | 0bab | 94ba886988ab98998ac9cbaa977abaa9899bbb9c |
| 38 | 047 | 7888477774666864888 8b99877744688a98a968c |
| 39 | 04 | 7998448444687776848898a967977687778698bb |
| 40 | 0 | 4968477477887777488999887969989a488b98b |

Table 6.3.1: $d_{i,j}$, the distance between STS(15)s, part II

| | 4444444444555555555566666666667777777778<br>1234567890123456789012345678901234567890 |
|---|---:|
| 41 | 0988474774787476778898887888886 8a79899ab |
| 42 | 088668667889668888 6b9b9644888964a6b88b9 |
| 43 | 04898898979aa989abcba988a8ba9bb8abb74c |
| 44 | 0688689747987998ac87988899879a979947c |
| 45 | 0747646687477788b798448878678869888c |
| 46 | 044488887686766a9887466866647889ab9 |
| 47 | 04476974788788986878868897748 98a9b |
| 48 | 0477887667788b98847686688879a99bb |
| 49 | 077686774888aa984676866477688ab8 |
| 50 | 068877766a8aa6874677686888966ab |
| 51 | 0444444876b797466767789899a4bc |
| 52 | 047647747c7987888644688a7a8ab |
| 53 | 0474748884847886 8777a99887bd |
| 54 | 0464466778766867888878a97ac |
| 55 | 074766a4b987664666886888bc |
| 56 | 07477b7a767846644aa9887bc |
| 57 | 048697866668788 7688a87ba |
| 58 | 0987787986799777978998bc |
| 59 | 04a966688687888a8babcc |
| 60 | 0ba9966676446694778ba |
| 61 | 07a4b9baaabbccdddddf |
| 62 | 09689997869ba898bcd |
| 63 | 09a9888798b97a9aca |
| 64 | 08688878a99aabaac |
| 65 | 06464664898997aa |
| 66 | 04468776a89889a |
| 67 | 0466746a8969cb |
| 68 | 04747898999cb |
| 69 | 08649a8a68bc |
| 70 | 0488884a6ba |
| 71 | 068a7788ca |
| 72 | 08a6b68c9 |
| 73 | 0a9ba886 |
| 74 | 0898668 |
| 75 | 0a4aac |
| 76 | 0b9dc |
| 77 | 0bbb |
| 78 | 08b |
| 79 | 0c |
| 80 | 0 |

Table 6.3.2: $h_{i,j}$ for pairs of STS(15)s, part I

| | 0000000001111111111222222222233333333334<br>1234567890123456789012345678901234567890 | 4444444445555555555666666666677777777778<br>1234567890123456789012345678901234567890 |
|---|---|---|
| 01 | 04688.ca.e.c...7.............g......... | .......................................f |
| 02 | 044488688c886a6aa.c..ccca..cee......... | ..............................eh...... |
| 03 | 04486487a7648478cbcccca8c.bcce........ | ..e................f...........e....... |
| 04 | 044844474446886a8aa8886aa8c8bcbe..gbb | d...ded....dcdfh.fcdc..e..d......f.c.... |
| 05 | 04446477468868ab.c8a88bba.8bc..e.f.. | .c.ef...d............e......d..ch....g. |
| 06 | 0486647774c8488acbbbb88c8k8cc....bb | .cccfcchd.c..ce.ebcg..ecdf...c.d..c.e... |
| 07 | 08a77.678.48a..f..o...b.......c... | ....f..............c.....d.....c...eccf |
| 08 | 0448744444488..7764777a7abb.c.caa | b.ccccbdd.cbcacb.cab..ccd....bd..fec.... |
| 09 | 044744487484aa444666477788bb.daa | ac.bbbabbbbaaadada7baaaacddbcbdd.bb8cf.. |
| 10 | 04444487477aa4644777747a8ab.bbb | acdbabcbbbcbaaadca9bacabccabcaccd.bbec.. |
| 11 | 04774c774466667744747477a8ea88 | 8a88ab8babbba8abb88a8a7cacaac8cabb8bcbgc |
| 12 | 0777cb864664444884767 88ae.c88 | 8h.cbbacbcbaaaccc87c9ccbce..fb.d.cc8.... |
| 13 | 04484477cc774788687998eb.b.a | ba.edbad.badabdddcac..acbfcgdc.h.c.b.d.. |
| 14 | 07667.7cf7764888989bab....e | a..d....g..dac..cdaccccc...e.b.....ff... |
| 15 | 084447aa7777477877aab8abab | aaaaccbacabcccaccbbac.7.cabcgabb.daagd.. |
| 16 | 068cc....b7......dbc..... | .......................f................ |
| 17 | 0477fc9a7777abea99.ecd.. | eaaafaffeba.....b.eaffacfd.db.c..de....d |
| 18 | 074aa7777747a7977.b.bba | fcbabbaaacbaabbaab8a.a9cbabbc8ac.acc.... |
| 19 | 0668ba9977b797bcc..b.b | gbccac8eeacbcad.daca9c4ccc.fcab.a.abcaa. |
| 20 | 086777777777a744aacbaa | acgbaaaba.b8a8aaa8486da8acbac8aabaaaac.f |
| 21 | 04a888a888c8aa4a.844 | 4b.a884888aa77aca7ac678aacbababcc8a.cf.. |
| 22 | 08aaa8acae8888dda77 | 7bdbaa7baa884488848a44a7bbaabaaacde.cb.. |
| 23 | 04444477447497ba88 | 88a768688a9988a8a687.87b8888978ab877aa.. |
| 24 | 04444764444a7aa68 | 8887886888a768989877b888a888b79ab874a9c. |
| 25 | 044444766468d886 | 6a779889888888698877.a799a9aa8a9ba97caae |
| 26 | 04447477787.986 | 6a998a88a88986888889.a899baa97a9ca97cc.. |
| 27 | 04747477a4c888 | 68779a88869888896679e948998887988677aaaf |
| 28 | 0777744878668 | 8aa9688688678688a947c97789a8a478a897b8.. |
| 29 | 0874478aca88 | 6cc788a998978aaa8a69c668aba9b8abd.7aaa.g |
| 30 | 0444684e886 | 684488889a9868889879cb77888a8798b879977d |
| 31 | 0777c8a8aa | ab778b8cbabaaa99baa9.caac8cbb99a8697e88e |
| 32 | 06487a968 | 6884886868876688a847c6488689a478884877cd |
| 33 | 0487b888 | 887866886674768786778a94888884888897ab9b |
| 34 | 089b688 | 88a786868867668674e977688864888888a8c. |
| 35 | 08e474 | 4b867877987748a8a899f6669a9778a9a7dc98.. |
| 36 | 088a6 | 8a4488898a879a9999accb7a98b8a99bc8aab88c |
| 37 | 0bab | d4ea886a88bb98a98aca..aa977bbaa98bdeb.9. |
| 38 | 047 | 7888477774666864888eba877744688a98aa68f |
| 39 | 04 | 7aa84484446877768488.8a9679776877786b8.. |
| 40 | 0 | 4a6847747788777774 88aaa887a69989a488ba8. |

Table 6.3.2: $h_{i,j}$ for pairs of STS(15)s, part II

| | 4444444445555555555666666666677777777778 |
|---|---|
| | 1234567890123456789012345678901234567890 |
| 41 | 0b884747747874767788c88878888868a7989a.b |
| 42 | 08866866678896688886.cb9644888964d6c88cc |
| 43 | 0489889897aaa98aab.fa988b8ba9be8ccc74f |
| 44 | 0688689747987998bd8798889987aaa79947e |
| 45 | 0747646687477788c7a8448878678869888. |
| 46 | 044488887686766a9887466866647889acb |
| 47 | 04476a74788788c868788688a7748a8abe |
| 48 | 0477887667788d98847686688879a99.. |
| 49 | 077686774888aaa84676866477688ad8 |
| 50 | 068877766a8.a6874677686888b66af |
| 51 | 0444444876b7974667677898a9a4d. |
| 52 | 047647747d7a87888644688a7a8ab |
| 53 | 04747488848478868777a9a887bd |
| 54 | 0464466778766867888878a97a. |
| 55 | 074766b4b987664666886888d. |
| 56 | 07477c7a767846644ba9887.. |
| 57 | 0486e78666687887688a87ca |
| 58 | 0a87787a8679a77a78a98d. |
| 59 | 04c966688687888a8bbb.. |
| 60 | 0baa966676446694778ca |
| 61 | 07.4dcbaad.bd....d.. |
| 62 | 09689997869da898b.. |
| 63 | 09ba888798ca7a9aca |
| 64 | 08688878a99acbaae |
| 65 | 06464664898ba7d. |
| 66 | 04468776a89889a |
| 67 | 0466746a8a69cb |
| 68 | 047478a8999ce |
| 69 | 08649a8a68cc |
| 70 | 0488884a6ea |
| 71 | 068a7788ca |
| 72 | 08a6b68d9 |
| 73 | 0aada886 |
| 74 | 0898668 |
| 75 | 0a4aa. |
| 76 | 0bc.. |
| 77 | 0b.. |
| 78 | 08b |
| 79 | 0. |
| 80 | 0 |

# Chapter 7

# Configurations in triple systems

## 7.1 Introduction

For this chapter we consider configurations of a more general nature than as described in Chapter 1. We allow pairs to occur more than once in a design and we even allow multiple blocks. However, to deal with the added complication of duplicate blocks we can no longer identify a block with the set of points associated with it.

An *incidence structure* [4, Chapter I] is a triple $(V, \mathcal{B}, \iota)$, where $V$ and $\mathcal{B}$ are disjoint sets and $\iota \subseteq V \times \mathcal{B}$ is a binary relation between $V$ and $\mathcal{B}$. The elements of $V$ are called *points*, the elements of $\mathcal{B}$ are called *blocks* and $\iota$ is an *incidence relation.* If $(x, B) \in \iota$, we write $x \ \iota \ B$ and we say that $x$ and $B$ are incident, or $x$ is incident with $B$, or $B$ is incident with $x$. If $B$ is a block, we define $(B) = \{x \in V : x \ \iota \ B\}$; that is, $(B)$ the set of points which are incident with $B$.

Let $S = (V, \mathcal{B}, \iota)$ and $S' = (V', \mathcal{B}', \iota')$ be two incidence structures with $|V| = |V'|$ and $|\mathcal{B}| = |\mathcal{B}'|$. A bijection $\phi : V \cup \mathcal{B} \to V' \cup \mathcal{B}'$ is called an *isomorphism* if satisfies the conditions $\phi(V) = V'$, $\phi(\mathcal{B}) = \mathcal{B}'$ and $x \ \iota \ B \Leftrightarrow \phi(x) \ \iota' \ \phi(B)$ for all $x \in V$ and $B \in \mathcal{B}$. When an isomorphism from $S$ to $S'$ exists, we say that $S$ and $S'$ are isomorphic, or $S \cong S'$, or $S'$ is an isomorph of $S$. An *automorphism* of $S$ is an isomorphism from $S$ to itself. The set of all automorphisms of an incidence structure $S$ together with the operation of composition is known as the *full automorphism group* of $S$ and is denoted by $\mathrm{Aut}(S)$.

Let $R = (U, \mathcal{A}, \eta)$ and $S = (V, \mathcal{B}, \iota)$ be two incidence structures. We say that $R$ *is a substructure of* $S$ if $U \subseteq V$, $\mathcal{A} \subseteq \mathcal{B}$ and $\eta \subseteq \iota$, and that $R$ *occurs in* $S$ if there is an incidence structure $R' = (U', \mathcal{A}', \eta') \cong R$ such that $R'$ is a substructure of $S$.

An incidence structure is *simple* if $(B_1) \neq (B_2)$ whenever $B_1$ and $B_2$ are distinct blocks. In a simple incidence structure, $(V, \mathcal{B}, \iota)$, we can identify a block $B$ with $(B)$, $\iota$ with the relation $\in$ of set membership and the triple $(V, \mathcal{B}, \in)$ with the pair $(V, \{(B) : B \in \mathcal{B}\})$.

A *triple system* of order $v$ and *index* $\lambda$, $\mathrm{TS}(v, \lambda)$, is an incidence structure $(V, \mathcal{B}, \iota)$ with the following properties: (i) $V$ is a set of cardinality $v$, (ii) $|(B)| = 3$ for all $B \in \mathcal{B}$, (iii) for every pair $\{x, y\}$ of distinct elements of $V$ there are precisely $\lambda$ blocks $B$ such that $x \; \iota \; B$ and $y \; \iota \; B$. If we identify a simple $\mathrm{TS}(v, \lambda)$, $(V, \mathcal{B}, \iota)$, with the pair $(V, \{(B) : B \in \mathcal{B}\})$, the implied incidence relation being set membership, then a $\mathrm{TS}(v, 1)$ is an $\mathrm{STS}(v)$, exactly as defined in Chapter 1.

In the context of triple systems, a *configuration* is an incidence structure where every point is incident with at least one block and every block has precisely three points incident with it.

For our purpose, the points of a configuration or a triple system are non-negative integers with the usual ordering. Pairs are ordered reverse-lexicographically, and triples are ordered according to the pairs they contain. If $A = \{x, y\}$, $x < y$, and $A' = \{x', y'\}$, $x' < y'$, are distinct pairs, then $A < A'$ if $y < y'$, or $y = y'$ and $x < x'$. If $B = \{x, y, z\}$, $x < y < z$, and $B' = \{x', y', z'\}$, $x' < y' < z'$, are distinct triples, then $B < B'$ if $y < y'$, or $y = y'$ and $x < x'$, or $\{x, y\} = \{x', y'\}$ and $z < z'$. This leads to a natural way of representing a configuration $(V, \mathcal{B}, \iota)$. We list the triples $(B)$ for $B \in \mathcal{B}$ in some order, usually the order defined above, taking repetitions into account. Thus when we speak of a configuration $\mathcal{C}$, we regard $\mathcal{C}$ as a list of triples from which the corresponding incidence structure can be recovered by giving a unique name to each item (block) in the list and constructing the incidence relation in the obvious manner. For example, we would identify the list $A_4 = (\{0, 1, 2\}, \{0, 1, 2\})$ with the two-block configuration $(\{0, 1, 2\}, \{X, Y\}, \{0, 1, 2\} \times \{X, Y\})$, where $(X) = (Y) = \{0, 1, 2\}$. When represented as an incidence structure it is plain that $A_4$ has full automorphism group generated by permutations of $\{0, 1, 2\}$ and permutations of $\{X, Y\}$; therefore $|\mathrm{Aut}(A_4)| = 12$.

If $\mathcal{C}$ is a configuration, we denote the number of blocks in $\mathcal{C}$ by $b(\mathcal{C})$, the number of points by $p(\mathcal{C})$ and the set of points by $P(\mathcal{C})$. The *degree* of a point is the number of blocks incident with it. Also we write $\ell(\mathcal{C}, \{a, b\})$ for the number of blocks incident with both $a$ and $b$, and $\ell(\mathcal{C})$ for the maximum of $\ell(\mathcal{C}, \{a, b\})$ over all pairs $\{a, b\} \subseteq P(\mathcal{C})$. A *simple* configuration is one with no repeated blocks, and a *Steiner* configuration, $\mathcal{C}$, also has the property that $\ell(\mathcal{C}) = 1$. A *generating* configuration, or *generator* is a configuration where every point has degree at least two.

## 7.2 Enumeration of $n$-block configurations

In Appendix C we give a complete list of the configurations of up to 4 blocks. The numbering assigned to some of the configurations is standard [16, 34] and generating configurations are indicated by an asterisk in the column headed 'degrees'.

A *canonical labelling* of a configuration $\mathcal{C}$ is an assignment of the integers $\{0, 1, \ldots, |P(\mathcal{C})| - 1\}$ to the points of $\mathcal{C}$ such that in its list form $\mathcal{C}$ is minimal under the ordering described in the previous section. Canonical labellings can be determined by Miller's algorithm [55], our version being based on the description in section 4.2 of Colbourn & Rosa [16]. The blocks in Appendix C are those that arise from a canonical labelling. For brevity, brackets and commas have been omitted. Two configurations are isomorphic if and only if they have the same canonical labellings. Furthermore, the order of the automorphism group of a simple configuration is equal to the number of its canonical labellings. For a non-simple system, $\mathcal{C}$, the number of canonical labellings needs to be multiplied by $\prod \beta(T)!$, where the product is over the distinct triples, $T$, of points of $\mathcal{C}$ and $\beta(T)$ is the number of blocks $B$ of $\mathcal{C}$ such that $(B) = T$. Observe that in a canonically labelled configuration $\mathcal{C}$, $\ell(\mathcal{C})$ is always equal to $\ell(\mathcal{C}, \{0, 1\})$, and $\mathcal{C}$ is simple if and only if it contains precisely one block incident with the elements of $\{0, 1, 2\}$.

The list of configurations in Appendix C was created in the usual manner. The $n$-block configurations were generated by adding a block in every possible manner to every $(n-1)$-block configuration; the results were canonically labelled and then sorted to eliminate duplicates. To our knowledge there is no significantly more efficient way of enumerating the $n$-block configurations $\mathcal{C}$ with $\ell(\mathcal{C}) = 1$ (i.e. those configurations that can occur in Steiner triple systems) other than by actually generating and counting them. The same appears to be the case for $n$-block configurations with $\ell(\mathcal{C}) = \lambda$ for any given, fixed value of $\lambda$.

However, if there is no restriction on $\ell(\mathcal{C})$, then it is possible to obtain a count by the application of Polya's enumeration theorem (see, for example, Read [60]). Let $C(n)$ denote the number of pairwise non-isomorphic $n$-block configurations, and $C_{\text{simple}}(n)$ the corresponding number of simple configurations. If $\pi$ is a permutation of a given set of $r$ elements, we associate with it a formal expression, $s_1^{j_1} s_2^{j_2} \ldots s_r^{j_r}$, called the *cycle type* of $\pi$. Here, the $s_i$ are regarded as indeterminates and for $i = 1, 2, \ldots r$ the exponent $j_i$ of $s_i$ is the number of cycles of length $i$ in the cyclic representation of $\pi$. Denote by $S_p$ the symmetric group on $p$ points.

We start with the cycle types of the permutations of $S_p$ and we use them to compute the corresponding cycle types of $S_p^{(3)}$, the group of permutations induced by $S_p$ on the $\binom{p}{3}$ triples of points. Given a cycle type of $S_p$,

$$s = s_1^{j_1} s_2^{j_2} \ldots s_p^{j_p},$$

we determine how it transforms to a cycle type in $S_p^{(3)}$. The analysis that follows subdivides conveniently into six cases.

(i) All three elements of the triple are in the same cycle of length $i$, say. There are $\frac{1}{6}i(i-1)(i-2)$ triples distributed over $[\frac{1}{6}(i-1)(i-2)]$ cycles of length $i$, plus one cycle of length $\frac{1}{3}i$ if $i \equiv 0 \pmod 3$. The cycle can be chosen in $j_i$ ways.

(ii) Two elements are in the same cycle of length $i$ and the third element is in a different cycle of the same length. There are $\frac{1}{2}i^2(i-1)$ triples distributed over $\frac{1}{2}i(i-1)$ cycles of length $i$. The cycle containing two elements can be chosen in $j_i$ ways; the cycle containing one element can be chosen in $j_i - 1$ ways.

(iii) The three elements are in different cycles of the same length, $i$. There are $i^3$ triples distributed over $i^2$ cycles of length $i$. The cycles can be chosen in $\binom{j_i}{3}$ ways.

(iv) Two elements are in the same cycle of length $h$ and one element is in a cycle of a different length, $i$. There are $\frac{1}{2}h(h-1)i$ triples. If $h$ is odd, then the two elements in the $h$-cycle generate $\frac{1}{2}(h-1)$ cycles of length $h$. Combining each one with a cycle of length $i$ generates $(h,i)$ cycles of length $[h,i]$. Altogether we have $\frac{1}{2}(h-1)(h,i)$ cycles of length $[h,i]$. If $h$ is even, then the two elements in the

$h$-cycle generate $\frac{1}{2}h - 1 = [\frac{1}{2}(h-1)]$ cycles of length $h$ and one cycle of length $\frac{1}{2}h$. Combining with a cycle of length $i$ generates $[\frac{1}{2}(h-1)](h,i)$ cycles of length $[h,i]$ and $(\frac{1}{2}h, i)$ cycles of length $[\frac{1}{2}h, i]$. The cycles can be chosen in $j_h j_i$ ways.

(v) The three elements are in different cycles, two elements are in cycles of the same length, $h$, and one element is in a cycle of a different length, $i$. There are $h^2 i$ triples. The elements in the cycles of length $h$ generate $h$ cycles of length $h$. Combining with a cycle of length $i$ generates $h(h,i)$ cycles of length $[h,i]$. The cycles can be chosen in $\binom{j_h}{2} j_i$ ways.

(vi) The three elements are in cycles of different lengths, $g$, $h$ and $i$. There are $ghi$ triples distributed over $ghi/[g,h,i]$ cycles of length $[g,h,i]$. The cycles can be chosen in $j_g j_h j_i$ ways.

Using the standard result that the number of permutations in $S_p$ with cycle type $s_1^{j_1} s_2^{j_2} \dots s_p^{j_p}$ is

$$\frac{n!}{1^{j_1} j_1!\ 2^{j_2} j_2!\ \dots\ p^{j_p} j_p!},$$

we combine the six cases together, giving the *cycle index* of $S_p^{(3)}$,

$$\begin{aligned} Z(S_p^{(3)}) \;=\; & \frac{1}{p!} \sum_{s_1^{j_1} s_2^{j_2} \dots s_p^{j_p} \text{ in } S_p} \left( \frac{p!}{1^{j_1} j_1!\ 2^{j_2} j_2!\ \dots\ p^{j_p} j_p!} \right. \\ & \cdot \prod_{1 \le i \le p} s_i^{[(i-1)(i-2)/6] j_i} \cdot \prod_{1 \le i \le p,\ 3|i} s_{i/3}^{j_i} \\ & \cdot \prod_{1 \le i \le p} s_i^{i(i-1) j_i (j_i - 1)/2} \cdot \prod_{1 \le i \le p} s_i^{i^2 j_i (j_i-1)(j_i-2)/6} \\ & \cdot \prod_{1 \le h \ne i \le p,\ h \text{ odd}} s_{[h,i]}^{(h-1)(h,i) j_h j_i /2} \\ & \cdot \prod_{1 \le h \ne i \le p,\ h \text{ even}} s_{[h,i]}^{(h/2-1)(h,i) j_h j_i} s_{[h/2,i]}^{(h/2,i) j_h j_i} \\ & \cdot \left. \prod_{1 \le h \ne i \le p} s_{[h,i]}^{h(h,i) j_h (j_h - 1) j_i/2} \cdot \prod_{1 \le g < h < i \le p} s_{[g,h,i]}^{ghi j_g j_h j_i/[g,h,i]} \right), \end{aligned}$$

where the sum is over all the cycle types $s_1^{j_1} s_2^{j_2} \dots s_p^{j_p}$ of $S_p$. Finally, we replace each occurrence of $s_a^b$ by $f(x^a)^b$ , where $f(x) = 1 + x + x^2 + \dots + x^m$, to obtain our desired generating function,

$$
\begin{aligned}
F(x) \;=\; & \sum_{s_1^{j_1} s_2^{j_2} \ldots s_p^{j_p} \text{ in } S_p} \left( \frac{1}{1^{j_1} j_1!\; 2^{j_2} j_2! \;\ldots\; p^{j_p} j_p!} \right. \\
& \cdot \prod_{1 \le i \le p} f(x^i)^{([(i-1)(i-2)/6] + (i(i-1)/2 + i^2 (j_i - 2)/6)(j_i - 1)) j_i} \cdot \prod_{1 \le i \le p,\ 3|i} f(x^{i/3})^{j_i} \\
& \cdot \prod_{1 \le h \ne i \le p} f(x^{[h,i]})^{([(h-1)/2] + h(j_h - 1)/2)(h,i) j_h j_i} \\
& \left. \cdot \prod_{1 \le h \ne i \le p,\ 2|h} f(x^{[h/2,i]})^{(h/2,i) j_h j_i} \cdot \prod_{1 \le g < h < i \le p} f(x^{[g,h,i]})^{g h i j_g j_h j_i / [g,h,i]} \right).
\end{aligned}
$$

If we regard the exponent $k$ of the term $x^k$ in $f(x)$ as representing the multiplicity of a triple in the list form of a configuration, then by Polya's enumeration theorem the coefficient of $x^n$ in $F(x)$ is the number of pairwise non-isomorphic $n$-block configurations of at most $p$ points and with triples repeated at most $m$ times. Hence if $p \ge 3n$ and $m \ge n$, then $C(n)$ is the coefficient of $x^n$ in the generating function. Simple configurations are enumerated by setting $p = 3n$ and $m = 1$. We present some values of $C(n)$ and $C_{\text{simple}}(n)$ in Table 7.1.

## 7.3 Counting configurations in a triple system

If $S$ is a triple system and $\mathcal{C}$ is a configuration, we denote by $n(\mathcal{C}, S)$ the number of occurrences of $\mathcal{C}$ in $S$. If the system $S$ is fixed (or clear from the context), we abbreviate $n(\mathcal{C}, S)$ to $n(\mathcal{C})$. If the configuration is denoted by a subscripted upper-case (italic) letter, $X_i$, say, we usually write the corresponding subscripted lower-case letter, $x_i$, for $n(X_i)$. For convenience we write $n^*(\mathcal{C})$, or $n^*(\mathcal{C}, S)$, for $n(\mathcal{C}, S)\,|\mathrm{Aut}(\mathcal{C})|$ and $x_i^*$ for $x_i\,|\mathrm{Aut}(X_i)|$.

We now describe a powerful general method for obtaining explicit formulae for $n(\mathcal{C}, S)$ for the $k$-block configurations, $\mathcal{C}$, given the formulae for all $(k-1)$-block configurations. For this purpose the generating configurations are of particular relevance because of a theorem established by Horák, Phillips, Wallis and Yucas [42]: *The single block together with all $j$-block generating configurations for $j \le k$ form a generating set for the $k$-block configurations.* Equivalently, for any $k$-block configuration $\mathcal{C}$ which has at least one point of degree 1,

$$
n(\mathcal{C}, S) = Q(v, \lambda) + \sum_{\mathcal{G}} Q_{\mathcal{G}}(v, \lambda) n(\mathcal{G}, S), \tag{7.1}
$$

Table 7.1: Enumeration of configurations

| $n$ | $C(n)$ | $C_{\text{simple}}(n)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 4 | 3 |
| 3 | 16 | 12 |
| 4 | 93 | 66 |
| 5 | 652 | 445 |
| 6 | 6369 | 4279 |
| 7 | 79568 | 53340 |
| 8 | 1256425 | 846254 |
| 9 | 24058631 | 16333946 |
| 10 | 543204998 | 371976963 |
| 11 | 14138916124 | 9763321109 |
| 12 | 417362929209 | 290473143807 |
| 13 | 13798729189578 | 9674133467729 |
| 14 | 505990335048034 | 357177322891321 |
| 15 | 20415765544541866 | 14503958827502886 |
| 16 | 900364519682003919 | 643502334799711633 |
| 17 | 43155049922002494115 | 31018731336031551119 |
| 18 | 2236988329443856718604 | 1616523352051185316626 |
| 19 | 124862936181977439454012 | 90689288905913623412837 |
| 20 | 7476052709321753156375756 | 5456178840303106057314759 |
| 21 | 478506183522725779096476581 | 350830170593891706361540379 |
| 22 | 32638841238874891261354722405 | 24035053807242494313138130137 |

where $\mathcal{G}$ runs through all the generating configurations of $j$ blocks, $j \leq k$, and the coefficients $Q(v, \lambda)$ and $Q_{\mathcal{G}}(v, \lambda)$ are polynomials in $v$ and $\lambda$. However, we believe that the starred functions are the more natural. We find that (7.1) takes on a significantly tidier form if it is written as

$$n^*(\mathcal{C}, S) = Q'(v, \lambda) + \sum_{\mathcal{G}} Q'_{\mathcal{G}}(v, \lambda) n^*(\mathcal{G}, S),$$

and, as we shall see, the functions $n^*(\mathcal{C}, S)$ and $n^*(\mathcal{G}, S)$ arise naturally in the proof of the main result of this section, which we state as follows.

**Theorem 7.3.1** *Let $S$ be a TS$(v, \lambda)$, let $k \geq 2$ be a positive integer and let $\mathcal{X}$ be a $(k-1)$-block configuration with point set $\{0, 1, \ldots, w-1\}$, where $w \leq v-3$. Let $\{a, b\}$ be two distinct points of $\{0, 1, \ldots, w-1, w+1, w+2\}$. For $i = 0, 1, \ldots, w$, $i \neq a, b$, denote by $\mathcal{Y}_i$ the $k$-block configuration obtained by adjoining a block incident with all elements of $\{a, b, i\}$ to $\mathcal{X}$. Let $\alpha$ denote the number of occurrences of the* *pair* $\{a,b\}$ *in* $\mathcal{X}$ *and let* $\delta = |\{a,b\} \cap \{w+1, w+2\}|$. *Then*

$$(\lambda - \alpha)\frac{(v-w)!}{(v-w-\delta)!}\,|\mathrm{Aut}(\mathcal{X})|\,n(\mathcal{X},S) = \sum_{\substack{i=0\\ i\neq a,b}}^{w} |\mathrm{Aut}(\mathcal{Y}_i)|\,n(\mathcal{Y}_i,S). \tag{7.2}$$

Before proving Theorem 7.3.1 we illustrate it with a couple of simple examples. Let $\mathcal{X}$ be the triangle configuration, $B_5$, $(\{0,1,2\},\{0,3,4\},\{1,3,5\})$, with full automorphism group order 6. Choose $a = 2$, $b = 4$. The configurations obtained by adding a block incident with points 2 and 4 are as follows: $\mathcal{Y}_0 = \mathcal{X} + \{2,4,0\}$, configuration $C_{34}$, $|\mathrm{Aut}(C_{34})| = 2$; $\mathcal{Y}_1 = \mathcal{X} + \{2,4,1\}$, $C_{35}$, $|\mathrm{Aut}(C_{35})| = 2$; $\mathcal{Y}_3 = \mathcal{X} + \{2,4,3\}$, $C_{35}$ again; $\mathcal{Y}_5 = \mathcal{X} + \{2,4,5\}$, $C_{16}$, $|\mathrm{Aut}(C_{16})| = 24$; $\mathcal{Y}_6 = \mathcal{X} + \{2,4,6\}$, $C_{14}$, $|\mathrm{Aut}(C_{14})| = 4$. Also $w = 6$, $\alpha = 0$ and $\delta = 0$. Thus according to the theorem we have

$$6\lambda b_5 = 2c_{34} + 2c_{35} + 2c_{35} + 24c_{16} + 4c_{14},$$

or

$$\lambda b_5^* = c_{34}^* + c_{35}^* + c_{35}^* + c_{16}^* + c_{14}^*.$$

On the other hand suppose $a = 2$ and $b$ is a point not in $\mathcal{X}$. Then $b = w + 1 = 7$ and the configurations obtained by adding a block incident with 2 and 7 are: $\mathcal{Y}_0 = \mathcal{X} + \{2,7,0\}$, $C_{48}$, $|\mathrm{Aut}(C_{48})| = 1$; $\mathcal{Y}_1 = \mathcal{X} + \{2,7,1\}$, $C_{48}$ again; $\mathcal{Y}_3 = \mathcal{X} + \{2,7,3\}$, $C_{15}$, $|\mathrm{Aut}(C_{15})| = 6$; $\mathcal{Y}_4 = \mathcal{X} + \{2,7,4\}$, $C_{14}$, $|\mathrm{Aut}(C_{14})| = 4$; $\mathcal{Y}_5 = \mathcal{X} + \{2,7,5\}$, $C_{14}$ again; $\mathcal{Y}_6 = \mathcal{X} + \{2,7,6\}$, $C_{12}$, $|\mathrm{Aut}(C_{12})| = 4$. Now $\delta = 1$ and

$$6\lambda(v-w)b_5 = c_{48} + c_{48} + 6c_{15} + 4c_{14} + 4c_{14} + 4c_{12},$$

or

$$\lambda(v-w)b_5^* = c_{48}^* + c_{48}^* + c_{15}^* + c_{14}^* + c_{14}^* + c_{12}^*.$$

**Proof of Theorem 7.3.1.** The theorem is proved by counting isomorphisms. First observe that the number of isomorphisms $\phi$ that map a fixed configuration $\mathcal{C}$ to a configuration $\phi(\mathcal{C})$ which is a substructure of $S$ is given by $n^*(\mathcal{C},S)$.

Let $\phi$ be an isomorphism from $\mathcal{X}$ to $\overline{\mathcal{X}} = \phi(\mathcal{X})$, where $\overline{\mathcal{X}}$ is a substructure of $S$. Suppose $a$ and $b$ are points of $\mathcal{X}$. Then the number of ways of choosing a new block incident with all elements of $\{\phi(a),\phi(b),j\}$ to append to $\phi(\mathcal{X})$ is precisely $\lambda - \alpha$ since there are altogether $\lambda$ blocks incident with both elements of $\{\phi(a),\phi(b)\}$

in $S$ of which $\alpha$ have already been accounted for in $\phi(\mathcal{X})$. If $a$ is in $\mathcal{X}$ but $b$ is a new point, then $\phi(b)$ can be chosen from any of the remaining $v-w$ points of the $\mathrm{TS}(v,\lambda)$. Also $\alpha = 0$ since the elements of $\{a,b\}$ are not both incident with any block of $\mathcal{X}$. Hence the number of ways of choosing a new block to append to $\phi(\mathcal{X})$ is now $\lambda(v-w)$. Similarly, if both $a$ and $b$ are new points, the number of ways of choosing a new block is $\lambda(v-w)(v-w-1)$ and, as before, $\alpha = 0$.

Combining the three cases together yields $(\lambda-\alpha)\frac{(v-w)!}{(v-w-\delta)!}$. Hence the number of ways of creating a substructure of $S$ consisting of $\overline{\mathcal{X}} = \phi(\mathcal{X})$, for some isomorphism $\phi$, plus an additional block incident with both elements of $\{\phi(a),\phi(b)\}$ is precisely

$$(\lambda-\alpha)\frac{(v-w)!}{(v-w-\delta)!}n^*(\mathcal{X},S).$$

We claim that this process counts exactly once every isomorphism $\phi$ for which $\phi(\mathcal{Y}_i)$ is a substructure of $S$ for some $\mathcal{Y}_i$, $i = 0,1,\ldots,w$.

Suppose $\phi(\mathcal{Y}_i)$ is a substructure of $S$. Then there is a point, $j$, of $S$ such that all elements of $\{\phi(a),\phi(b),j\}$ are incident with a block of $S$ and if $j$ is also a point of $\phi(\mathcal{X})$, it is clear that $\mathcal{Y}_i$ is obtained from $\mathcal{X}$ by appending the block $\{a,b,i\}$, where $i = \phi^{-1}(j)$. On the other hand, if $j$ is not a point of $\phi(\mathcal{X})$, then $i = w$, $j = \phi(w)$ and $\mathcal{Y}_w$ is obtained from $\mathcal{X}$ by appending the block incident with all elements of $\{a,b,w\}$. Hence if $\phi$ is an isomorphism from $\mathcal{Y}_i$ to $\overline{\mathcal{Y}_i}$ and $\overline{\mathcal{Y}_i}$ is a substructure of $S$, then $\overline{\mathcal{Y}_i}$ is precisely $\phi(\mathcal{X})$ augmented by a block incident with the elements of $\{\phi(a),\phi(b),\phi(i)\}$.

Now suppose that $\overline{\mathcal{Y}_i} \cong \mathcal{Y}_i$, and $\overline{\mathcal{Y}_i}$ is a substructure of $S$. Suppose $\overline{\mathcal{Y}_i}$ is created in two ways, (i) by adding a block incident with the elements of $\{\phi(a),\phi(b),\phi(i)\}$ to $\phi(\mathcal{X})$, or (ii) by adding a block incident with the elements of $\{\psi(a),\psi(b),\psi(i)\}$ to $\psi(\mathcal{X})$. Then $\phi(\mathcal{Y}_i) = \psi(\mathcal{Y}_i) = \overline{\mathcal{Y}_i}$. So $\gamma : x \mapsto \phi^{-1}(\psi(x))$ is an automorphism of $\mathcal{Y}_i$. But $\gamma$ is the identity function if and only if $\phi = \psi$. $\square$

In [34], Grannell, Griggs & Mendelsohn initiated the study of generating sets and bases for configurations. In particular, they obtained formulae for $c_1, c_2, \ldots, c_{15}$, the frequencies of occurrence of the 4-block configurations that can occur in an $\mathrm{STS}(v)$, as linear functions of 1 and $c_{16}$ (with polynomials in $v$ as coefficients). This work was continued by Danziger, Mendelsohn, Grannell & Griggs in [17] to the 5-block

Steiner configurations.

As an immediate application of Theorem 7.3.1 we extend the results of [17] by listing the 6-block Steiner triple configurations in Appendix A and giving in Appendix B the formulae for their frequencies of occurrence in Steiner triple systems. As a check, we can confirm that the right-hand sides of the 277 expressions in Appendix B do indeed sum to $\binom{v(v-1)/6}{6}$.

Also we extend the results of [34] to general triple systems, $\mathrm{TS}(v,\lambda)$. The generating set is somewhat larger and the formulae depend on $\lambda$ as well as $v$. Here it is considerably tidier to display fractionless expressions, and to produce them we recall the notation $x_i^* = |\mathrm{Aut}(X_i)|\, x_i$, where $x_i$ is the frequency of occurrence of $X_i$ in the $\mathrm{TS}(v,\lambda)$. Therefore our set of generating counts for the 4-block configurations is

$$\{1, a_4^*, b_6^*, b_{16}^*, c_{16}^*, c_{17}^*, c_{18}^*, c_{19}^*, c_{20}^*, c_{21}^*, c_{22}^*, c_{23}^*, c_{72}^*, c_{73}^*, c_{89}^*, c_{90}^*\}.$$

The formulae are presented in Appendix D. We use the starred notation from which the usual form of the counts may be recovered on dividing by the appropriate automorphism group orders, as listed in Appendix C. Then substituting $\lambda = 1$ reproduces the considerably simpler formulae of [34] for Steiner configurations. As a further check, for each $k$, one can add together the (unstarred) counts for all $k$-block configurations and confirm that they sum to $\binom{\lambda v(v-1)/6}{k}$.

With Theorem 7.3.1 it is possible to push the computations considerably further. However, in the limited space of this thesis it is not feasible to display all our results. Although we do make use of some of the formulae for 6-block configurations in Steiner triple systems, we find that most of the expressions we have obtained are quite complicated and therefore possibly of questionable application. For instance, the expression for $g_{17100}$ (eight disjoint blocks) is over 3000 characters long even when restricted to $\lambda = 1$.

# Chapter 8

# Existentially closed graphs

## 8.1 Introduction

The *block intersection graph* of a Steiner triple system $(V, \mathcal{B})$ is the graph whose set of points is $\mathcal{B}$ and for $A, B \in \mathcal{B}$, $A \sim B$ if and only if $A \cap B$ is non-empty. A *strongly regular graph* $\mathrm{SRG}(v, k, \lambda, \mu)$ is a regular graph of degree $k$ on $v$ vertices with the property that every pair of adjacent vertices has $\lambda$ common neighbours and every pair of non-adjacent vertices have $\mu$ common neighbours. It is easy to see from the structure of Steiner triple systems that the block intersection graph of an $\mathrm{STS}(v)$ is an $\mathrm{SRG}(v(v-1)/6, 3(v-3)/3, (v+3)/2, 9)$.

A graph is *n-existentially closed* if for every set $S$ of $n$ vertices, for every subset $T$ of $S$ there exists $x \notin S$ such that for every $y \in T$, $x \sim y$, and for every $z$ in $S \setminus T$, $x \not\sim z$. Erdős and Rényi [21] proved the interesting result that for any fixed value of $n$, almost all graphs are $n$-existentially closed. However, relatively few specific examples of $n$-existentially closed graphs are known for $n \geq 2$. One possibility, which we investigate in this chapter, is that 2- and 3-existentially closed graphs appear as block intersection graphs of Steiner triple systems. Indeed, we will show that every $\mathrm{STS}(v)$ with $v \geq 13$ has a 2-existentially closed block intersection graph. Furthermore, we obtain a characterization of those Steiner triple systems whose block intersection graphs are 3-existentially closed, which implies that they can exist only for a limited range of orders of $\mathrm{STS}(v)$, possibly just $v = 19$. However, these graphs do exist and we are able to identify two 3-existentially closed strongly regular graphs with 57 vertices, an order for which such a graph seems to have been previously unknown.

In Table 8.1, we list some specific configurations that we will refer to in the rest of the chapter. We give them their standard names and their canonical labellings in the point set $\{0, 1, \dots\}$. As a matter of notational convenience, in Table 8.1 and elsewhere we omit set brackets and commas from the description of the pairs and blocks of a configuration; thus, for example, the block $\{x, y, z\}$ is denoted simply by $xyz$. The significance of the fourth column, headed '$\mu$', is explained in section 8.3 (Definition 8.3.1).

Table 8.1: Configurations

| Symbol | Name | Blocks | $\mu$ |
|---|---|---|---|
| $A_1$ | | $\{012, 345\}$ | 15 |
| $A_2$ | | $\{012, 034\}$ | $\infty$ |
| $B_1$ | | $\{012, 345, 678\}$ | 15 |
| $B_2$ | hut | $\{012, 034, 567\}$ | 15 |
| $B_3$ | 3-star | $\{012, 034, 056\}$ | $\infty$ |
| $B_4$ | 3-path | $\{012, 034, 156\}$ | 15 |
| $B_5$ | triangle | $\{012, 034, 135\}$ | $\infty$ |
| $C_6$ | | $\{012, 034, 135, 678\}$ | 15 |
| $C_9$ | 4-path | $\{012, 034, 156, 378\}$ | 15 |
| $C_{13}$ | | $\{012, 034, 156, 278\}$ | 15 |
| $C_{14}$ | | $\{012, 034, 135, 246\}$ | 14 |
| $C_{16}$ | Pasch | $\{012, 034, 135, 245\}$ | $\infty$ |
| $D_1$ | mitre | $\{012, 034, 135, 236, 456\}$ | 13 |
| $D_2$ | | $\{012, 034, 135, 236, 146\}$ | $\infty$ |
| $D_{15}$ | | $\{012, 034, 135, 246, 078\}$ | 14 |
| $D_{19}$ | | $\{012, 034, 156, 357, 468\}$ | 15 |
| $E_1$ | semihead | $\{012, 034, 135, 236, 146, 245\}$ | $\infty$ |
| $E_2$ | | $\{012, 034, 135, 236, 147, 567\}$ | 13 |
| $E_3$ | 6-cycle | $\{012, 034, 135, 246, 257, 367\}$ | 13 |
| $E_7$ | | $\{012, 034, 135, 236, 146, 247\}$ | 13 |
| $E_8$ | | $\{012, 034, 135, 236, 147, 257\}$ | 13 |
| $E_{12}$ | | $\{012, 034, 135, 236, 147, 058\}$ | 14 |
| $E_{28}$ | | $\{012, 034, 135, 246, 257, 168\}$ | 14 |
| $E_{33}$ | | $\{012, 034, 135, 246, 567, 078\}$ | 14 |

## 8.2 Systems with 2-e-c block intersection graphs

**Theorem 8.2.1** *The block intersection graph of a Steiner triple system of order $v$ is 2-existentially closed if and only $v \geq 13$.*

**Proof.** Consider an STS$(v)$ with $v \geq 13$. There are two 2-block configurations that can occur in the STS($v$); namely (i) a pair of intersecting blocks, $A_2 : \{abc, ade\}$, or (ii) a pair of parallel blocks, $A_1 : \{abc, def\}$. For each of these configurations we need to show that there exists (a) a block that intersects both blocks of the configuration, (b) a block that intersects exactly one block of the configuration, and (c) a block that is parallel to both blocks of the configuration. We consider each case in turn.

Case (i)(a). There are at least three blocks of the STS($v$) that contain the point $a$. Hence there exists a suitable block, $afg$, say, that intersects both $abc$ and $ade$.

Case (i)(b). Since $v \geq 9$ there are at least four blocks that contain the point $b$, and therefore at least one of them must avoid the points $a$, $d$ and $e$. Hence there exists a block that intersects with $abc$ but not with $ade$. (This is impossible if the system is an STS(7). Hence the block intersection graph of the STS(7) is not 2-existentially closed)

Case (i)(c). Choose a point, $h$, say, outside the configuration $\{abc, ade\}$. Since $v \geq 13$ there are at least six blocks that contain $h$, and therefore at least one of them must avoid blocks $abc$ and $ade$. (On the other hand, in an STS(9) any block containing $h$ also intersects the configuration $\{abc, ade\}$, and therefore its block intersection graph is not 2-existentially closed.)

Case (ii)(a). It is clear that there exists a block that intersects both $abc$ and $def$; for instance, the block that contains the pair $ad$.

Case (ii)(b). Since $v \geq 13$, there are at least five blocks containing $a$, and hence at least one of them must avoid block $def$.

Case (ii)(c). Choose a point, $i$, say, outside the configuration $\{abc, def\}$. If $v \geq 15$, there are at least seven blocks containing $i$, and hence there must be a block that avoids both $abc$ and $def$. If $v = 13$, we argue as follows. Partition the points of the STS(13) into two sets, $A$ and $B$, where $A = \{a, b, c, d, e, f\}$ and $B$ consists of the remaining seven points. Let $\nu(\alpha, \beta)$ denote the number of blocks that contain precisely $\alpha$ points of $A$ and $\beta$ points of $B$. Clearly $\nu(3, 0) = 2$ and $\nu(2, 1) = 9$. Hence to account for the 42 $AB$ pairs we must have $\nu(1, 2) = (6 \cdot 7 - 2 \cdot 9)/2 = 12$. Therefore $\nu(0, 3) = 26 - 2 - 9 - 12 = 3$ and each of these three blocks will be disjoint from $\{abc, def\}$.

We have also proved that the block intersection graphs of the STS(7) and the STS(9) are not 2-existentially closed. □

## 8.3 Systems with 3-e-c block intersection graphs

We now consider the existence of Steiner triple systems having 3-existentially closed block intersection graphs. Since an $n$-existentially closed graph is $(n-1)$-existentially closed, we can eliminate the cases STS(7) and STS(9). Also by a straightforward computation we have established that neither the two STS(13)s nor the 80 STS(15)s have 3-existentially closed block intersection graphs.

The next two lemmas are used in the proof of Theorem 8.3.1.

**Lemma 8.3.1** *Let $v \geq 19$ and suppose $\mathcal{X}$ is a three-block configuration in a Steiner triple system of order $v$. Let $x$ be a point in $\mathcal{X}$. Then there exists a block that intersects each of the blocks of $\mathcal{X}$ that contain $x$ and none of the blocks of $\mathcal{X}$ that do not contain $x$.*

**Proof.** Since $v \geq 19$, there are at least nine blocks containing $x$. Also $\mathcal{X}$ has at most six points not in the same block as $x$, Hence there exists at least one block which intersects all the blocks of $\mathcal{X}$ that contain $x$ and is disjoint from all the blocks of $\mathcal{X}$ that do not contain $x$. □

**Lemma 8.3.2** *Let $v \geq 19$ and suppose $\mathcal{X}$ is a three-block configuration in a Steiner triple system of order $v$. Suppose also that $\mathcal{X}$ has at most eight points. Then there exists a block that is disjoint from $\mathcal{X}$.*

**Proof.** Let $y$ be a point in the STS($v$) which is not in $\mathcal{X}$. Then, since $v \geq 19$, there exist at least nine blocks containing $y$ and, since $\mathcal{X}$ contains at most eight points, there is at least one block (containing $y$) that is disjoint from $\mathcal{X}$. □

**Theorem 8.3.1** *For $v \geq 19$, the block intersection graph of a Steiner triple system of order $v$, STS($v$), is 3-existentially closed if and only if all of the following criteria are satisfied:*

*(i) the STS($v$) contains no sub-STS(7);*

*(ii) the STS($v$) contains no sub-STS(9);*

*(iii) for every set of three parallel blocks, there exists a block which intersects all three.*

**Proof.** First we deal with the 'if' part of the theorem. The proof is similar to that of Theorem 8.2.1 but with a somewhat larger number of cases to check.

There are five 3-block configurations that can occur in the STS($v$); namely (i) three intersecting blocks, a 3-star, $B_3 : \{abc, ade, afg\}$; (ii) a triangle, $B_5 : \{abc, ade, bdf\}$; (iii) a 3-path, $B_4 : \{abc, ade, bfg\}$; (iv) a 2-path and a disjoint block (also known as a 'hut'), $B_2 : \{abc, ade, fgh\}$; (v) three disjoint blocks, $B_1 : \{abc, def, ghi\}$. For each of these five configurations we need to show that there exists (a) a block that intersects all three blocks of the configuration; (b) blocks that intersect two given blocks of the configuration but not the third; (c) blocks that intersects a given block of the configuration but not the other two; and (d) a block that is parallel to all three blocks of the configuration. We consider each case separately.

Lemma 8.3.1 immediately deals with cases (i)(a), (i)(c), (ii)(b), (ii)(c), (iii)(c), (iv)(c) and (v)(c), and Lemma 8.3.2 with (i)(d), (ii)(d), (iii)(d) and (iv)(d).

Case (i)(b). Without loss of generality, consider the two blocks $abc$ and $ade$. The set of pairs $\{bd, be, cd, ce\}$ yields a set, $\mathcal{F}$, of four blocks. If all four blocks of $\mathcal{F}$ intersect $afg$, the third block of the $B_3$, then $\{abc, ade, afg\} \cup \mathcal{F}$ is an STS(7). Since this possibility is ruled out by criterion (i), there exists a block in $\mathcal{F}$ that intersects $abc$ and $ade$ but avoids $afg$.

Case (ii)(a). The block containing the pair $af$ intersects all three blocks of the triangle ($B_5$).

Case (iii)(a). The block containing the pair $af$ intersects all three blocks of the 3-path ($B_4$).

Case (iii)(b). For pairs of blocks $\{abc, ade\}$ and $\{abc, bfg\}$, Lemma 8.3.1 ensures that there exist blocks $A$ and $B$ such that $a \in A$, $b \in B$ and $A \cap bfg = B \cap ade = \emptyset$. The only other case is the pair of blocks $\{ade, bfg\}$. Here we note that the blocks generated by the pairs $df$ and $dg$ cannot both intersect block $abc$.

Case (iv)(a). The block containing the pair $af$ intersects all three blocks of the $B_2$.

Case (iv)(b). For the pair of blocks $\{abc, ade\}$, Lemma 8.3.1 ensures that there exists a block which contains $a$ and avoids $fgh$. For the pair $\{abc, fgh\}$ we note that at least one of the blocks generated by the pairs $\{bf, bg, bh\}$ must be disjoint from block $ade$. The pair $\{ade, fgh\}$ is handled similarly.

Case (v)(a). This is given by criterion (iii) in the statement of the theorem.

Case (v)(b). Without loss of generality we may consider just the pair of blocks $abc$ and $def$. Let $\mathcal{G}$ be the set of blocks generated by the nine pairs $ad$, $ae$, $af$, $bd$, $be$, $bf$, $cd$, $ce$ and $cf$. If each of these blocks intersect $ghi$, the third block of the $B_1$, then $\{abc, def, ghi\} \cup \mathcal{G}$ is an STS(9), contrary to criterion (ii). Hence at least one block of $\mathcal{G}$ is disjoint from $ghi$.

Case (v)(d). If $v \geq 21$, an argument like that in the proof of Lemma 8.3.2 shows that there exists a block which is disjoint from the $B_1$ configuration. So let $v = 19$ and partition points of the STS(19) into two sets, $A$ and $B$, where $a = \{a, b, c, d, e, f, g, h, i\}$. Let $\nu(\alpha, \beta)$ denote the number of blocks in the STS(19) which contain $\alpha$ points of $A$ and $\beta$ points of $B$. Let $\nu(3, 0) = n$. Then by a simple computation we have $\nu(2, 1) = 36 - 3n$, $\nu(1, 2) = 9 + 3n$ and $\nu(0, 3) = 12 - n$. Hence $n \leq 12$. However, $n = 12$ implies that $A$ is the point set of an STS(9), which is ruled out by hypothesis (ii). Therefore $n \leq 11$ and $\nu(0, 3) \geq 1$, as required.

That completes the first part of the proof. For the other implication, suppose that $S$ has a 3-existentially closed block intersection graph. Then observe that criterion (iii) follows trivially, and the arguments given in cases (i)(b) and (v)(b), above, are reversible, implying that $S$ cannot have a sub-STS(7) or a sub-STS(9).

□

In the sequence of theorems which follows we determine an upper limit of $v$ for which an STS($v$) can have a 3-existentially closed block intersection graph. The results of this section have been published with somewhat different proofs in Forbes, Grannell & Griggs [28].

Logically our next theorem is not necessary, for in Theorem 8.3.3 we prove a stronger result. However, we include Theorem 8.3.2 because the proof is very simple.

**Theorem 8.3.2** *if $v \geq 37$, a Steiner triple system of order $v$ cannot have a 3-existentially closed block intersection graph.*

**Proof.** We consider the frequencies of occurrence in an STS($v$) of the configurations $B_1$ and $C_{13}$. Let the frequencies be denoted by $b_1$ and $c_{13}$, respectively, and let $c_{16}$ denote the frequency of occurrence of the Pasch configuration ($C_{16}$). Then

$$b_1 = v(v-1)(v-3)(v-7)(v^2-19v+96)/1296 \tag{8.1}$$

and

$$c_{13} = v(v-1)(v-3)(v^2-18v+85)/48 - 4c_{16} \tag{8.2}$$

(Colbourn & Rosa [16], Grannell, Griggs & Mendelsohn [34]). Since a specific instance of a $C_{13}$ configuration arises from a unique $B_1$ by adding a block that intersects all three blocks of the $B_1$, criterion (iii) of Theorem 8.3.1 fails if $b_1 < c_{13}$. This is indeed the case for $v \geq 37$. □

**Definition 8.3.1** *Let $\mathcal{C} \subseteq \mathcal{D}$ be configurations. For $X, Y \in \mathcal{C}$, let*

$$\mathcal{L}_{\mathcal{D},X,Y} = \{X, Y\} \cup \{\{x, y, z\} \in \mathcal{D} : x \in X, y \in Y\}.$$

*Define*

$$\mu(\mathcal{C}) = \min\{p(\mathcal{L}_{\mathcal{C},X,Y}) + 11 - b(\mathcal{L}_{\mathcal{C},X,Y})\},$$

*where the minimum is taken over all pairs of parallel blocks $X, Y \in \mathcal{C}$. Define $\mu(\mathcal{C}) = \infty$ if $\mathcal{C}$ does not contain a pair of parallel blocks. If $S = (V, \mathcal{B})$ is a Steiner triple system, define $\mu(S) = \mu(\mathcal{B})$.*

A few remarks are in order. Let $S = (V, \mathcal{B})$ be a Steiner triple system which contains a configuration $\mathcal{C}$. Then $\mu(\mathcal{C})$ is the minimum number of points of $S$ generated by a pair of parallel blocks in $\mathcal{C}$ together with the nine cross-links, on the assumption that cross-links which are not in $\mathcal{C}$ give rise to distinct points in $S$. Thus

$$\mu(\mathcal{B}) = \min\{p(\mathcal{L}_{\mathcal{C},X,Y}) + 11 - b(\mathcal{L}_{\mathcal{C},X,Y}) : X, Y \in \mathcal{B}, X \cap Y = \emptyset\} \leq \mu(\mathcal{C}).$$

Suppose $\{X, Y\}$ is a pair of parallel blocks. In some of the following theorems we are interested in $p(\mathcal{L}_{\mathcal{B},X,Y})$. In the absence of any further information the best upper bound for $p(\mathcal{L}_{\mathcal{B},X,Y})$ is 15, and this is reflected in the value $\mu(\{X,Y\}) = 15$. But if, for example, we know that $S$ contains a mitre configuration, $D_1$, we can do a little better. For if $X$ and $Y$ are the parallel blocks of the mitre, then $p(\mathcal{L}_{\mathcal{B},X,Y}) \leq 13$, and indeed we have $\mu(D_1) = 13$. The constant 11 in the definition was chosen to make this work. The points of $\{X,Y\}$ together with the cross-links (blocks of the form $\{x,y,z\}$ with $x \in X$ and $y \in Y$) which occur in $\mathcal{L}_{\mathcal{B},X,Y}$ contribute $p(\mathcal{L}_{\mathcal{B},X,Y})$ points to $\mu$, and there are a further $9 - (b(\mathcal{L}_{\mathcal{B},X,Y}) - 2)$ points arising from cross-links that do not occur in $\mathcal{L}_{\mathcal{B},X,Y}$.

We note for future reference the $\mu$ values of a number of configurations. These are given in column 4 of Table 8.1 (page 104).

**Lemma 8.3.3** *If $\mathcal{C} \subseteq \mathcal{D}$ are configurations, then $\mu(\mathcal{D}) \leq \mu(\mathcal{C})$.*

**Proof.** This is obvious. □

**Lemma 8.3.4** *If $S$ is an STS(7)-free STS($v$) which contains a Pasch configuration, then $\mu(S) \leq 13$.*

**Proof.** A Pasch configuration in $S$ extends immediately to a $D_2$ configuration, $\{abc, ade, bdf, cdg, beg\}$, say. Since $S$ is STS(7)-free, we cannot have $a*g = c*e = f$. (Recall that $\alpha*\beta$ is the third point in the block containing $\alpha$ and $\beta$.) If $a*g = x$, say, is different from $f$, then $\{abc, ade, bdf, cdg, beg, agx\}$ is an $E_7$. Similarly if $c*e \neq f$. The result follows because $\mu(E_7) = 13$. □

**Lemma 8.3.5** *Let $S = (V, \mathcal{B})$ be an STS($v$) with maximum independent set of cardinality $m$. If $\mu(S) + m < v$, then $S$ cannot have a 3-existentially closed block intersection graph.*

**Proof.** By definition, $S$ contains disjoint blocks $X$ and $Y$ such that $\mathcal{L}_{\mathcal{B},X,Y}$ has $\mu(S)$ points. Let $W = V \setminus P(\mathcal{L}_{\mathcal{B},X,Y})$. Then $|W| \geq v - \mu(S)$ and, since this is greater than $m$, there exists a block $Z \subseteq W$. Furthermore, none of the blocks that intersect both $X$ and $Y$ also intersect $Z$. Thus criterion (iii) in the statement of Theorem 8.3.1 fails. □

**Lemma 8.3.6** *If $S$ is an STS($v$) with maximum independent set of cardinality at most $v - 15$, then $S$ cannot have a 3-existentially closed block intersection graph.*

**Proof.** We can assume that $S$ does not contain an STS(7). The number of $C_{14}$ configurations in an STS($v$) is given by

$$c_{14} = \frac{1}{4}v(v-1)(v-3) - 6c_{16},$$

where $c_{16}$ is the number of Pasch configurations, [35]. So $c_{14} = 0$ implies $c_{16} = v(v-1)(v-3)/24$, and this can only occur if $v = 2^{n+1} - 1$ and the STS($v$) is the point-line design of the projective geometry PG$(n, 2)$. But these systems contain sub-STS(7)s. Therefore $S$ contains at least one $C_{14}$ configuration. Since $\mu(C_{14}) = 14$, the result follows from Lemma 8.3.5. □

**Theorem 8.3.3** *If $v \geq 31$, a Steiner triple system of order $v$ cannot have a 3-existentially closed block intersection graph.*

**Proof.** An STS($v$) cannot have an independent set of cardinality greater than $(v+\epsilon)/2$, where $\epsilon = 1$ if $v \equiv 3$ or $7 \pmod{12}$, $\epsilon = 0$ otherwise. Hence if $v \geq 31$, the largest possible independent set in an STS($v$) satisfies the requirements of Lemma 8.3.6. □

**Lemma 8.3.7** *If $\mathcal{C}$ is a Pasch-free configuration with 15 blocks and 12 points, then $\mu(\mathcal{C}) \leq 13$.*

**Proof.** This is the result of a straightforward computation. □

**Lemma 8.3.8** *If $\mathcal{C}$ is a Pasch-free configuration with 19 blocks and 13 points, then $\mu(\mathcal{C}) \leq 13$.*

**Proof.** Suppose $\mathcal{C}$ is a 19-block, 13-point, Pasch-free configuration with $\mu(\mathcal{C}) \geq 14$. Then $\mathcal{C}$ has a point $p$ of degree at most 4, implying that $\mathcal{C}$ contains a Pasch-free configuration $\mathcal{D}$ with at least 15 blocks, at most 12 points and $\mu(\mathcal{D}) \geq 14$. This contradicts Lemma 8.3.7. □

**Lemma 8.3.9** *If $\mathcal{C}$ is a Pasch-free configuration with 23 blocks and 14 points, then $\mu(\mathcal{C}) \leq 13$.*

**Proof.** Suppose $\mathcal{C}$ is a 23-block, 14-point, Pasch-free configuration with $\mu(\mathcal{C}) \geq 14$. Then $\mathcal{C}$ has a point of degree at most 4. Hence $\mathcal{C}$ contains a Pasch-free configuration $\mathcal{D}$ with at least 19 blocks, at most 13 points and $\mu(\mathcal{D}) \geq 14$, contradicting Lemma 8.3.8. $\square$

**Theorem 8.3.4** *A Steiner triple system of order 27 cannot have a 3-existentially closed block intersection graph.*

**Proof.** Let $S$ be an STS(27), let $I$ be an independent set in $S$ of maximum cardinality and let $\mathcal{J}$ be the configuration consisting of all those blocks of $S$ which have no points in $I$.

If $|I| = 14$, then $\mathcal{J}$ is the block set of an STS(13) and by a straightforward computation we have $\mu(\mathcal{J}) = 12$; hence the result follows from Lemma 8.3.5. If $|I| \leq 12$, we apply Lemma 8.3.6. Hence we can assume that $|I| = 13$.

If $\mu(S) \leq 13$, we use Lemma 8.3.5. If $\mu(S) \geq 14$, we can assume by Lemma 8.3.4 that $S$ is also Pasch-free. Thus $\mathcal{J}$ is a Pasch-free configuration with 26 blocks, 14 points and $\mu(\mathcal{J}) \geq 14$. By Lemma 8.3.9 this is impossible. $\square$

**Definition 8.3.2** *Denote by $\Gamma$ the set of configurations $\mathcal{G}$ such that $\mathcal{G}$ has 22 blocks and 13 points, $\mathcal{G}$ is STS(7)-free and $\mu(\mathcal{G}) \geq 13$.*

**Lemma 8.3.10** *For each $\mathcal{G} \in \Gamma$, there exist two configurations $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{G}$ such that*

*(i) for $i = 1, 2$, $\mathcal{C}_i = \mathcal{L}_{\mathcal{G},X_i,Y_i}$, where $X_i, Y_i$ are disjoint blocks of $\mathcal{G}$;*

*(ii) $\mu(\mathcal{C}_1) = \mu(\mathcal{C}_2) = 13$;*

*(iii) $P(\mathcal{C}_1) \neq P(\mathcal{C}_2)$;*

*(iv) $|P(\mathcal{C}_1) \cap P(\mathcal{C}_2)| \geq 8$.*

**Proof.** A straightforward computation shows that $|\Gamma| = 90$ and that each $\mathcal{G} \in \Gamma$ has the stated property. $\square$

**Lemma 8.3.11** *Let $S$ be an STS(25), let $I$ be an independent set of maximum cardinality in $S$ and let $\mathcal{J}$ be the configuration consisting of all those blocks of $S$ which have no points in $I$. Then $S$ cannot have a 3-existentially closed block intersection graph except possibly when $\mu(S) = 13$ and $\mathcal{J} \in \Gamma$.*

**Proof.** We know that $\mu(S) \leq 14$ and $|I| \leq 12$. If $\mu(S) \leq 12$, or if $\mu(S) = 13$ and $|I| \leq 11$, or if $|I| \leq 10$, the result follows from Lemma 8.3.5.

Suppose $\mu(S) = 14$ and $|I| = 11$ or 12. By Lemma 8.3.4 we can assume also that $S$ is Pasch-free. If $|I| = 11$, $\mathcal{J}$ has 23 blocks and 14 points, contradicting Lemma 8.3.9. If $|I| = 12$, $\mathcal{J}$ has 22 blocks and 13 points, contradicting Lemma 8.3.8.

Thus we may assume that $\mu(S) = 13$ and $|I| = 12$. Hence $\mathcal{J} \in \Gamma$. □

**Lemma 8.3.12** *Let $I_1$ and $I_2$ be distinct independent sets of cardinality 12 in an STS(25). Then $|I_1 \cap I_2| \leq 6$.*

**Proof.** Put $w = 0$ in part (i) of Theorem 4.5.2. □

**Theorem 8.3.5** *There does not exist an STS(25) with a 3-existentially closed block intersection graph.*

**Proof.** Let $S = (V, \mathcal{B})$ be an STS(25) with a 3-existentially closed block intersection graph. We know from Lemma 8.3.11 that $S$ contains a configuration $\mathcal{G} \in \Gamma$. Consider the configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfying conditions (i)–(iv) of Lemma 8.3.10. For $i = 1, 2$, let $\{X_i, Y_i\}$ be the pairs of disjoint blocks in the definition of $\mathcal{C}_i$, let $\mathcal{D}_i = \mathcal{L}_{\mathcal{B},X_i,Y_i}$ and let $I_i = V \setminus P(\mathcal{D}_i)$. Since $\mu(S) = \mu(\mathcal{C}_1) = \mu(\mathcal{C}_2) = 13$, we have $\mu(\mathcal{D}_1) = \mu(\mathcal{D}_2) = 13$. Also $P(\mathcal{D}_1) = P(\mathcal{D}_2) = 13$, $P(\mathcal{D}_1) \neq P(\mathcal{D}_2)$ and $|P(\mathcal{D}_1) \cap P(\mathcal{D}_2)| \geq 8$. Therefore $|I_1| = |I_2| = 12$ and $|I_1 \cap I_2| \geq 7$. Furthermore, as in the proof of Lemma 8.3.5, $I_1$ and $I_2$ are independent sets. But by Lemma 8.3.12, $I_1$ and $I_2$ cannot exist. □

We are not prepared to state any results or make any conjectures regarding Steiner triple systems of order 21.

Finally, by trial we have found two Steiner triple systems of order 19 which have 3-existentially closed block intersection graphs. The first is the cyclic STS(19) generated from starter blocks {018, 025, 04a}, which has full automorphism group of order 57. It is the same as the one denoted by #A3 in the listing of [53]. There is a picture of the block intersection graph on the front cover of the mathematics magazine *M500* [52]. The second system has full automorphism group of order 8 and consists of the set of 57 blocks

{012, 034, 135, 236, 146, 245, 057, 068, 569, 178, 279, 37a, 47b, 67c, 28d, 38e, 48a, 58f, 09f, 19g, 39c, 49h, 89i, 0ae, 1ab, 2ah, 5ai, 6ag, 9ad, 0bg, 2bf, 3bi, 5bd, 6bh, 8bc, 9be, 0ch, 1cd, 2ci, 4ce, 5cg, acf, 0di, 3dg, 4df, 6de, 7dh, 1ef, 2eg, 5eh, 7ei, 3fh, 6fi, 7fg, 4gi, 8gh, 1hi}.

We are unaware of any other Steiner triple system with this property. However, we know that if there exists a third STS(19), it must have trivial full automorphism group and it must contain at least one Pasch configuration. Indeed we have tested all 164758 STS(19)s with non-trivial automorphism group as well as the 2591 anti-Pasch STS(19)s, which were kindly made available by Petteri Kaski and Patric Östergård of the Helsinki University of Technology. Also we have tried to construct STS(19)s with 3-existentially closed block intersection graphs with the help of a computer, and in fact this is how we initially discovered the above system with automorphism group of order 8. But no others appeared.

The computer methods which we applied to the STS(19)s failed to identify any STS(21) with a 3-existentially closed block intersection graph. In fact we have not been able to find an STS(21) whose block intersection graph is in any reasonable sense 'close' to being 3-existentially closed.

## 8.4 STS(25) and STS(27) revisited

In this section we prove a theorem which shows that if $S$ is an STS(25) or an STS(27) and $S$ has a 3-existentially closed block intersection graph, then $\mu(S) \leq 13$. The result can be used to eliminate the purely computational Lemma 8.3.7 from the proofs of Theorems 8.3.4 and 8.3.5.

**Theorem 8.4.1** *Let $S$ be an STS(7)-free Steiner triple system of order 25 or 27 with $\mu(S) \geq 14$. Then there exists in $S$ three parallel blocks such that no block of $S$ intersects all three of them.*

**Proof.** By Lemmas 8.3.3 and 8.3.4, the condition $\mu(S) \geq 14$ implies that $S$ is Pasch-free and mitre-free. Furthermore, since $\mu(E_2) = \mu(E_3) = 13$, $S$ also cannot contain either of the configurations $E_2$ and $E_3$. In the terminology of Chapter 9, $S$ is 6-sparse.

We collect together formulae for the numbers of occurrences in $S$ of some of the configurations listed in Table 8.1. As is usual, if the configuration is denoted by the subscripted upper-case letter $X_i$, the count is denoted by the corresponding lower-case letter, $x_i$. We have

$$d_{15} = 27c_{16} + 3d_1 - 9n_v/8 - 3vc_{16} + n_v v/8,$$

$$d_{19} = 6c_{16} + 2d_1 - 5n_v/6 + vn_v/12,$$

$$e_{12} = -12c_{16} + 8e_1 + n_v/6,$$

$$e_{28} = -12c_{16} - e_2 + n_v/2,$$

$$e_{33} = -6c_{16} - 3d_1 + n_v/4,$$

where $n_v = v(v-1)(v-3)$. They appear in Danziger, Mendelsohn, Grannell & Griggs [17] and (for $e_{12}$, $e_{28}$ and $e_{33}$) Appendix B of this thesis. Furthermore, $\mu(S) \geq 14$ implies that $c_{16} = d_1 = e_1 = e_2 = 0$. Therefore, also making use of (8.1) and (8.2), if $S$ is an STS(27),

$$b_1 = 81120,\ c_{13} = 115128,\ d_{15} = 37908,\ d_{19} = 23868,$$

$$e_{12} = 2808,\ e_{28} = 8424,\ e_{33} = 4212,$$

and if $S$ is an STS(25),

$$b_1 = 45100,\ c_{13} = 71500,\ d_{15} = 26400,\ d_{19} = 16500,$$

$$e_{12} = 2200,\ e_{28} = 6600,\ e_{33} = 3300.$$

We split up the $B_1$ configurations of $S$ as follows.

Let $\zeta$ denote the number of $B_1$ configurations that have no cross-links (that is, there are no blocks of $S$ that intersect all three blocks of the $B_1$).

Let $\alpha_1$ denote the number of $B_1$ configurations that have one cross-link.

Let $\beta_1$ denote the number of $B_1$ configurations that have two parallel cross-links.

Let $\beta_2$ denote the number of $B_1$ configurations that have two intersecting cross-links.

Let $\gamma_1$ denote the number of $B_1$ configurations that have three parallel cross-links.

Let $\gamma_2$ denote the number of $B_1$ configurations that have three cross-links, two of which intersect and are disjoint from the third. (i.e. the cross-links by themselves form a $B_2$ configuration.)

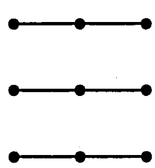
Let $\gamma_3$ denote the number of $B_1$ configurations that have three cross-links which form a $B_4$ configuration (3-path).

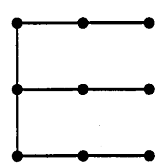
Let $\gamma_4$ denote the number of $B_1$ configurations that have three cross-links which form a $B_5$ configuration (triangle).

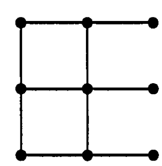
Let $\delta_1$ denote the number of $B_1$ configurations that have four cross-links which form a $C_{13}$ configuration (three parallel blocks and one block intersecting them all).

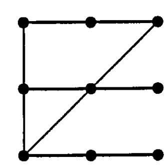
Let $\delta_2$ denote the number of $B_1$ configurations that have four cross-links which form a $C_6$ configuration (a triangle and a disjoint block).

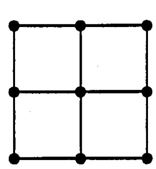
Let $\delta_3$ denote the number of $B_1$ configurations that have four cross-links which form a $C_9$ configuration (4-path).

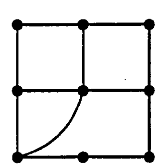

$\zeta$

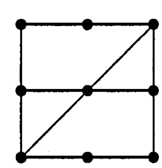

$\alpha_1$

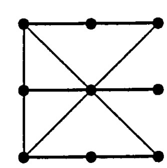

$\beta_1$

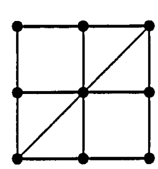

$\beta_2$

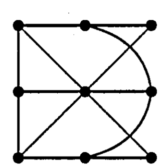

$\gamma_1$

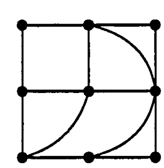

$\gamma_2$


$\gamma_3$


$\gamma_4$


$\delta_1$


$\delta_2$


$\delta_3$

Since $\mu(S) \geq 14$ and $S$ is Pasch-free, $S$ contains no other types of $B_1$ configuration. Indeed, it is easy to see that the configuration consisting of the cross-links must have at most nine points of which none is of degree 3 and at most three are of degree 2.

We wish to show that $\zeta > 0$. Clearly,

$$b_1 = \zeta + \alpha_1 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \delta_1 + \delta_2 + \delta_3.$$

By considering the number of cross-links in each type of $B_1$ configuration,

$$c_{13} = \alpha_1 + 2(\beta_1 + \beta_2) + 3(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + 4(\delta_1 + \delta_2 + \delta_3).$$

By considering the number of intersecting pairs of cross-links,

$$d_{15} = \beta_2 + \gamma_2 + 2\gamma_3 + 3\gamma_4 + 3(\delta_1 + \delta_2 + \delta_3),$$

and by considering the number of disjoint pairs of cross-links,

$$d_{19} = \beta_1 + 3\gamma_1 + 2\gamma_2 + \gamma_3 + 3(\delta_1 + \delta_2 + \delta_3).$$

Similarly, by counting triangles amongst the cross-links,

$$e_{12} = \gamma_4 + \delta_2,$$

by considering 3-paths in the cross-links,

$$e_{28} = \gamma_3 + 3\delta_1 + 2\delta_3,$$

and by considering $B_2$ configurations in the cross-links,

$$e_{33} = \gamma_2 + 3\delta_2 + 2\delta_3.$$

Combining these formulae, we have

$$b_1 - c_{13} + d_{15} + \frac{2d_{19}}{3} - e_{12} - \frac{2e_{28}}{3} - \frac{e_{33}}{2} = \zeta - \frac{\beta_1}{3} - \frac{\gamma_2}{6} - \frac{\delta_2}{2} - \frac{\delta_3}{3}.$$

Making use of the counts given above, we can compute the left-hand side of this last equality. If $S$ is an STS(25), it is 2750; if $S$ is an STS(27), it is 9282. Hence in both cases $\zeta > 0$, as required. □

# Chapter 9

# 6-sparse Steiner triple systems

## 9.1 Introduction

In 1976, Erdös [20] conjectured that for every integer $k \geq 4$, for all sufficiently large admissible $v$, there exists an STS($v$) with the property that it contains no configurations having $n$ blocks and $n+2$ points for any $n$ satisfying $4 \leq n \leq k$. Such an STS($v$) is said to be *$k$-sparse*. Clearly, a $k$-sparse system is also $k'$-sparse for every $k'$ satisfying $4 \leq k' \leq k$. A possible reason that configurations having two more points than blocks form the subject of the conjecture lies in the next two theorems.

**Theorem 9.1.1** *Suppose that $n \geq 2$ and that $v$ is admissible with $v \geq n+3$. Then any STS($v$) contains a configuration having $n$ blocks and $n+3$ points.*

**Proof.** This is Theorem 1.1 of [25]. □

**Theorem 9.1.2** *For every integer $d \geq 3$ and for every integer $n$ satisfying $n \geq \lceil \frac{d}{2} \rceil$ there exists $v_0(n,d)$ such that for all admissible $v \geq v_0(n,d)$, every STS($v$) contains a configuration having $n$ blocks and $n+d$ points.*

**Proof.** This is Corollary 1.1.1 of [25]. □

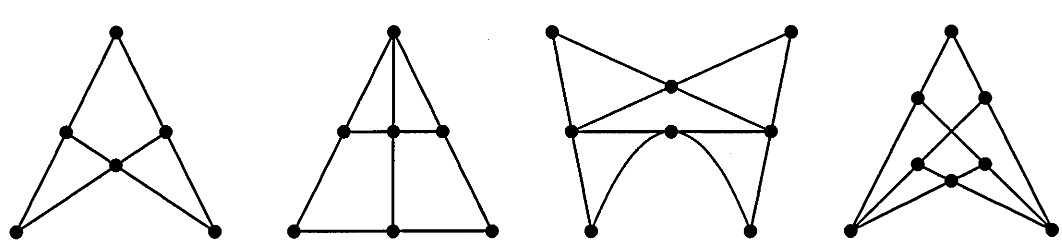
In Table 9.1 we list all those configurations which have $n$ blocks and $n+2$ points for $n = 4, 5, 6$. From this table it can be seen that there is only one configuration having four blocks and six points, namely the Pasch configuration. The existence of 4-sparse (better known as anti-Pasch) STS($v$)s for all admissible $v$, except $v = 7$ and 13, was established in a series of papers [8, 38, 48, 36].

| $n$ | Name | Blocks | |
|---|---|---|---|
| 4 | $C_{16}$ (Pasch) | 012, 034, 135, 245 | |
| 5 | $D_1$ (mitre) | 012, 034, 135, 236, 456 | |
| 5 | $D_2$ | 012, 034, 135, 236, 146 | contains Pasch |
| 6 | $E_2$ (crown) | 012, 034, 135, 236, 147, 567 | |
| 6 | $E_3$ (6-cycle) | 012, 034, 135, 246, 257, 367 | |
| 6 | $E_6$ | 012, 034, 135, 236, 146, 057 | contains Pasch |
| 6 | $E_7$ | 012, 034, 135, 236, 146, 247 | contains Pasch |
| 6 | $E_8$ | 012, 034, 135, 236, 147, 257 | contains mitre |

Table 9.1: Configurations having $n$ blocks and $n + 2$ points.

Some progress has also been made with 5-sparse systems. There is only one Pasch-free configuration having five blocks and seven points, namely the mitre ($D_1$). Thus a system is 5-sparse if and only if it is anti-Pasch and anti-mitre. In a sequence of publications [15, 47, 31] and culminating in the recent papers by Fujiwara [32] and Wolfe [71], it is established that anti-mitre systems exist for all admissible orders, apart from $v = 9$. Systems which are 5-sparse are known for $v \equiv 1, 19 \pmod{54}$ and for many sporadic values outside these residue classes [47, 32].

There are precisely two Pasch-free and mitre-free configurations with six blocks and eight points, namely $E_2$ (crown) and $E_3$ (6-cycle) as shown in Table 9.1. Thus a system is 6-sparse if and only if it contains no Pasch configurations, no mitres, no 6-cycles and no crowns. Up to the time of writing [25], no non-trivial 6-sparse systems were known.



Pasch ($C_{16}$) mitre ($D_1$) crown ($E_2$) 6-cycle ($E_3$)

In the next section we give a construction method for block transitive Steiner triple systems as used by Grannell, Griggs & Murphy [35] to produce nine new perfect Steiner triple systems. Using this construction we discovered 29 6-sparse systems with orders ranging from $v = 139$ to $v = 4447$.

## 9.2 Block transitive Steiner triple systems

The following theorem is equivalent to a result in [35].

**Theorem 9.2.1** *Suppose that $v$ is a prime congruent to 7 modulo 12 and that $\chi$ is a multiplicative character of $GF(v)$ of order 6. Suppose also that $\alpha \in GF(v)$ has the property that $\chi(\alpha) \neq -1, 0, 1$ and that $\chi(1-\alpha)\chi(\alpha) = \pm 1$. Let $G$ denote the group comprising all mappings on $GF(v)$ having the form $x \to ax + b$ for $a, b \in GF(v)$ with $\chi(a) = 1$. Then the orbit generated by the block $\{0, 1, \alpha\}$ under the action of $G$ forms a block transitive $STS(v)$.*

**Proof.** See [35]. □

In what follows we will refer to a system constructed in this fashion as a *block transitive* STS($v$) *with parameter* $\alpha$, tacitly assuming that $v \equiv 7 \pmod{12}$ is prime and that arithmetic is performed in GF($v$). For each value of $\alpha$, we examined the systems for 6-sparseness, taking advantage of block transitivity to simplify the work. Indeed, the test for an STS($v$), $(V, \mathcal{B})$, is an $O(v)$ process. We only need to examine the $3(v-3)$ triangles whose points of degree two are $\{a, b, c\}$, where $(a, b) \in \{(0, 1), (0, \alpha), (1, \alpha)\}$ and $c \in V \setminus \{0, 1, \alpha\}$.

A further simplification is obtained by observing that, if $\alpha$ satisfies the conditions of the theorem then, as shown in [35], the six STS($v$)s generated by the blocks $\{0, 1, \beta\}$ for $\beta \in \{\alpha, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, 1-\frac{1}{\alpha}, \frac{1}{\alpha}\}$ are all isomorphic. This observation reduces the number of cases to be checked. Furthermore, if $\chi(2) = 1$ then the STS($v$) with parameter $\alpha$ contains the Pasch configuration $\{\{0, 1, \alpha\}, \{0, 2, 2\alpha\}, \{1, 2, \alpha + 1\}, \{\alpha, \alpha + 1, 2\alpha\}\}$ and so cannot be 6-sparse.

The results of a computer search are collected in Table 9.2. Altogether we have found 29 6-sparse block transitive Steiner triple systems produced by the construction, including two non-isomorphic STS(139)s and two non-isomorphic STS(3259)s. We remark that the system with $v = 139$ and $\alpha = 51$ is isomorphic to the perfect block transitive STS(139) given in [35]. To construct that STS(139) the value $\alpha = 25$ was used, and this is related to ours by $51 \equiv 1 - 1/25 \pmod{139}$. The search was exhaustive for $v \leq 9\,150\,625$. In section 9.4 we will show that there is no need to search beyond this limit. None of the systems in Table 9.2 is 7-sparse.

| $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ |
|---|---|---|---|---|---|---|---|
| 139 | 51 | 907 | 68 | 1303 | 971 | 2707 | 1837 |
| 139 | 118 | 967 | 210 | 1531 | 42 | 3259 | 562 |
| 151 | 37 | 991 | 76 | 1699 | 506 | 3259 | 1286 |
| 463 | 261 | 1039 | 356 | 2083 | 800 | 3319 | 511 |
| 523 | 501 | 1051 | 660 | 2179 | 1820 | 4447 | 210 |
| 571 | 528 | 1087 | 519 | 2311 | 1593 | | |
| 691 | 468 | 1171 | 931 | 2503 | 1287 | | |
| 859 | 616 | 1291 | 833 | 2539 | 180 | | |

Table 9.2: Block transitive 6-sparse Steiner triple systems.

Next we prove some elementary lemmas used by Theorem 9.3.1, our main 6-sparse-preserving recursive construction.

**Lemma 9.2.1** *Let $S = (V, \mathcal{B})$ be a block transitive STS$(v)$ with parameter $\alpha$. Then (i) $\alpha \notin \{0, 1, -1, 2, \frac{1}{2}\}$ and (ii) $\alpha^2 \notin \{-1, 2\alpha - 2, \alpha - \frac{1}{2}\}$.*

**Proof.** For (i), there is in each case either a block which does not have three distinct points or a pair which appears in more than one block. For (ii), since $v \equiv 3 \pmod 4$, $-1$ is not a quadratic residue modulo $v$, and consequently $\alpha^2+1 = 0$, $\alpha^2-2\alpha+2 = 0$ and $2\alpha^2 - 2\alpha + 1 = 0$ are not solvable in GF$(v)$. □

**Lemma 9.2.2** *Let $S = (V, \mathcal{B})$ be a block transitive STS$(v)$ with parameter $\alpha$. Suppose $\{x, y, z\}$ and $\{\mu x, \mu y, \mu z\}$ are blocks of $S$. Then either $\chi(\mu) = 1$ or $\alpha^2 = \alpha - 1$.*

**Proof.** We may assume that $x = q, y = p + q, z = p\alpha + q$, and that

$$\{\mu x, \mu y, \mu z\} = \{s, r + s, r\alpha + s\} \tag{9.1}$$

where $p, q, r, s \in$ GF$(v)$ and $\chi(p) = \chi(r) = 1$. We examine each of the six permutations of (9.1). Taking first the case when $\mu x = s$ and $\mu y = r + s$, we have $(\mu y - \mu x)/(y - x) = r/p$, so $\mu = r/p$ and hence $\chi(\mu) = 1$. In the remaining five cases we compute $\mu$ in two ways from the ratios $(\mu y - \mu x)/(y - x)$ and $(\mu z - \mu x)/(z - x)$.

This yields the following implications.

$$
\begin{aligned}
\mu x = s,\ \mu y = r\alpha + s \quad &\Rightarrow \quad \alpha^2 = 1,\\
\mu x = r + s,\ \mu y = s \quad &\Rightarrow \quad \alpha = \frac{1}{2},\\
\mu x = r + s,\ \mu y = r\alpha + s \quad &\Rightarrow \quad \alpha^2 = \alpha - 1,\\
\mu x = r\alpha + s,\ \mu y = s \quad &\Rightarrow \quad \alpha^2 = \alpha - 1,\\
\mu x = r\alpha + s,\ \mu y = r + s \quad &\Rightarrow \quad \alpha = 2.
\end{aligned}
$$

If $\alpha^2 = 1$ then $\alpha = \pm 1$, and these values together with the values $\frac{1}{2}$ and 2 are excluded by Lemma 9.2.1. □

**Lemma 9.2.3** *Let $S = (V, \mathcal{B})$ be a block transitive STS$(v)$ with parameter $\alpha$. Suppose that $\mu \neq 0$ and that $\{c, d, g\}$ and $\{b, e, h\}$ are blocks of $S$. Then the two equalities*

$$b - e = (d - c)\mu \quad \textit{and} \quad h - b = (d - g)\mu \tag{9.2}$$

*cannot hold simultaneously unless*

$$\alpha^2 \in \{1 - \alpha, \alpha + 1, 3\alpha - 1\}. \tag{9.3}$$

**Proof.** Assume that $\{c, d, g\} = \{q, p + q, p\alpha + q\}$ and $\{b, e, h\} = \{s, r + s, r\alpha + s\}$ where $p, q, r, s \in GF(v)$ and $\chi(p) = \chi(r) = 1$. Given these representations of the blocks $\{c, d, g\}$ and $\{b, e, h\}$, we refer to the coefficient of $p$ for $c$, $d$ and $g$, and the coefficient of $r$ for $b$, $e$ and $h$, as the *type* of the point. The type is thus 0, 1 or $\alpha$.

For each of the 36 valid ways to combine the types of $c, d, g, b, e$ and $h$, we compute $\mu p/r$ in two ways, one for each of the equalities in (9.2). We may assume that either $c$ has type 0, or $c$ has type 1 and $g$ has type $\alpha$. For otherwise we make the two interchanges $c \leftrightarrow g$ and $e \leftrightarrow h$ (which involve pairs in the same block). Then (9.2) becomes $b - h = (d - g)\mu$ and $e - b = (d - c)\mu$, which is the same as (9.2) but with $\mu$ replaced by $-\mu$. Hence there are only 18 cases to consider.

We present the analysis of these cases in Table 9.3, which shows the two values of $\mu p/r$ (column 4) and their common solution, if any, for $\alpha$ (column 5) for each combination of the types of $c$, $d$, $g$ (column 2) and $b$, $e$, $h$ (column 3). It is straightforward to verify the contents of the table. In rows 1, 3, 11, 13 and 15, the expressions for $\mu p/r$ yield a contradiction. In rows 4, 5, 8, 9, 12, 14 and 18, the expression in column 5 contradicts Lemma 9.2.1. In the remaining cases, rows 2, 6, 7, 10, 16 and 17, each pair of expressions for $\mu p/r$ implies (9.3). $\square$

| | $cdg$ type | $beh$ type | $\mu p/r$ | solution |
|---|---|---|---|---|
| 1 | $01\alpha$ | $01\alpha$ | $-1,\ \alpha/(1-\alpha)$ | – |
| 2 | $01\alpha$ | $0\alpha 1$ | $-\alpha,\ 1/(1-\alpha)$ | $\alpha^2=\alpha+1$ |
| 3 | $01\alpha$ | $10\alpha$ | $1,\ -1$ | – |
| 4 | $01\alpha$ | $1\alpha 0$ | $1-\alpha,\ 1/(\alpha-1)$ | $\alpha^2=2\alpha-2$ |
| 5 | $01\alpha$ | $\alpha 01$ | $\alpha,\ 1$ | $\alpha=1$ |
| 6 | $01\alpha$ | $\alpha 10$ | $-1+\alpha,\ \alpha/(\alpha-1)$ | $\alpha^2=3\alpha-1$ |
| 7 | $0\alpha 1$ | $01\alpha$ | $-1/\alpha,\ \alpha/(\alpha-1)$ | $\alpha^2=1-\alpha$ |
| 8 | $0\alpha 1$ | $0\alpha 1$ | $-1,\ 1/(\alpha-1)$ | $\alpha=0$ |
| 9 | $0\alpha 1$ | $10\alpha$ | $1/\alpha,\ 1$ | $\alpha=1$ |
| 10 | $0\alpha 1$ | $1\alpha 0$ | $-1+1/\alpha,\ 1/(1-\alpha)$ | $\alpha^2=3\alpha-1$ |
| 11 | $0\alpha 1$ | $\alpha 01$ | $1,\ -1$ | – |
| 12 | $0\alpha 1$ | $\alpha 10$ | $(\alpha-1)/\alpha,\ \alpha/(1-\alpha)$ | $2\alpha^2=2\alpha-1$ |
| 13 | $10\alpha$ | $01\alpha$ | $1,\ -1$ | – |
| 14 | $10\alpha$ | $0\alpha 1$ | $\alpha,\ -1/\alpha$ | $\alpha^2=-1$ |
| 15 | $10\alpha$ | $10\alpha$ | $-1,\ -1+1/\alpha$ | – |
| 16 | $10\alpha$ | $1\alpha 0$ | $-1+\alpha,\ 1/\alpha$ | $\alpha^2=\alpha+1$ |
| 17 | $10\alpha$ | $\alpha 01$ | $-\alpha,\ (\alpha-1)/\alpha$ | $\alpha^2=1-\alpha$ |
| 18 | $10\alpha$ | $\alpha 10$ | $1-\alpha,\ 1$ | $\alpha=0$ |

Table 9.3: Lemma 9.2.3.

**Lemma 9.2.4** *Let $S=(V,\mathcal{B})$ be a block transitive STS($v$) with parameter $\alpha$. Suppose that $\mu\neq 0$ and that $\{a,g,h\}$ and $\{b,d,f\}$ are blocks of $S$ for which the two equalities*

$$b-d=(h-g)\mu \quad \textit{and} \quad b-f=(h-a)\mu \tag{9.4}$$

*hold simultaneously. Then either $\chi(\mu)=1$ or $\alpha^2=\alpha-1$.*

**Proof.** Let $b'=b+a\mu-f$, $d'=d+a\mu-f$, and $f'=a\mu$. By block transitivity, $\{b',d',f'\}$ is a block in $\mathcal{B}$. Furthermore, (9.4) implies $\{b',d',f'\}=\{a\mu,g\mu,h\mu\}$. Hence the result follows from Lemma 9.2.2. $\square$

## 9.3 Tripling and product constructions

**Theorem 9.3.1** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$, and $V = GF(v)$. Put $V' = V \times \{0,1,2\}$ and let*

$$\begin{aligned} \mathcal{B}' &= \{\{a_i, b_i, c_i\} : \{a,b,c\} \in \mathcal{B},\ i = 0,1,2\} \\ &\cup \{\{x_0, y_1, (x\beta + y\gamma)_2\} : x, y \in GF(v)\}, \end{aligned}$$

*where $\beta, \gamma \neq 0$ are fixed parameters in $GF(v)$. Then $S' = (V', \mathcal{B}')$ is a Steiner triple system of order $3v$. Furthermore*

*(i) if $S$ is anti-Pasch, then $S'$ is also anti-Pasch;*

*(ii) if $S$ is anti-mitre, $\alpha^2 \neq \alpha - 1$, and $\chi(\beta), \chi(\gamma), \chi(-\beta/\gamma) \neq 1$, then $S'$ is also anti-mitre;*

*(iii) if $S$ has no crowns and $\alpha^2 \notin \{1-\alpha, \alpha+1, 3\alpha-1\}$, then $S'$ also has no crowns;*

*(iv) if $S$ has no 6-cycles, $\alpha^2 \neq \alpha - 1$, and $\chi(\beta), \chi(\gamma), \chi(-\beta/\gamma) \neq -1$, then $S'$ also has no 6-cycles.*

*As a consequence, if $S$ is 6-sparse, $\alpha^2 \notin \{\alpha - 1, 1 - \alpha, \alpha + 1, 3\alpha - 1\}$, and $\chi(\beta), \chi(\gamma), \chi(\beta/\gamma) \neq \pm 1$, then $S'$ is also 6-sparse.*

**Proof.** It is worth remarking that the conditions on $\beta$ and $\gamma$ in the final sentence can be achieved, for example, by taking $\beta = \alpha$ and $\gamma = 1/\alpha$. It is easily verified that if $\beta$ and $\gamma$ are non-zero modulo $v$, the operation defined by $x \circ y = x\beta + y\gamma$ satisfies the axioms of a quasigroup. The construction itself is an application of the generalized direct product (see [16] page 39, for example), and so $S'$ is an STS($3v$).

If $x_i$ is a point of $V'$, we refer to $i$ as the *level* of $x_i$. We describe a block of $\mathcal{B}'$ as *horizontal* if all of its points have the same level; otherwise we describe it as *vertical.* A vertical block contains a point at each of the three levels 0, 1 and 2.

For each of the Pasch, mitre, crown, and 6-cycle configurations, we assume that $S$, but not $S'$, is free of that configuration, and we deduce a contradiction. The arguments are independent of each other; so, for instance, if $S$ contains Pasch configurations, mitres and 6-cycles but no crowns, we can still deduce that $S'$ is crown-free.

**Case (i) The Pasch configuration.**

Suppose $\mathcal{C}$ is a Pasch configuration in $S'$. It is easy to show that if $\mathcal{C}$ has a horizontal block, then all blocks of $\mathcal{C}$ are horizontal, contrary to the hypothesis that $S$ does not contain a Pasch configuration. Therefore $C$ has no horizontal blocks. Indeed, by exploiting the symmetry of the Pasch configuration, we can assume that

$$\mathcal{C} = \{\{a_0, b_1, c_2\}, \{a_0, e_1, d_2\}, \{f_0, b_1, d_2\}, \{f_0, e_1, c_2\}\}.$$

Then $c = a\beta + b\gamma = f\beta + e\gamma$ and $d = a\beta + e\gamma = f\beta + b\gamma$. Hence $(b-e)\gamma = (e-b)\gamma$ and therefore $b_1 = e_1$, a contradiction.

**Case (ii) The mitre.**

Suppose $\mathcal{D}$ is a mitre in $S'$. It is straightforward to verify that the number of horizontal blocks containing the apex (i.e. the unique point of degree 3 in this configuration) of $\mathcal{D}$ is either zero or one.

**Case (ii.a) No horizontal block contains the apex.**

The two disjoint blocks must be horizontal. We consider three cases according to the level of the apex.

If the apex has level 0, we can assume that

$$\mathcal{D} = \{\{a_0, b_1, e_2\}, \{a_0, c_1, f_2\}, \{a_0, d_1, g_2\}, \{b_1, c_1, d_1\}, \{e_2, f_2, g_2\}\}.$$

Then $e = a\beta + b\gamma$, $f = a\beta + c\gamma$, $g = a\beta + d\gamma$. By block transitivity, $\{b\gamma, c\gamma, d\gamma\} = \{e - a\beta, f - a\beta, g - a\beta\} \in \mathcal{B}$. But by Lemma 9.2.2 this implies $\chi(\gamma) = 1$, a contradiction.

If the apex has level 1, we can assume that

$$\mathcal{D} = \{\{a_1, b_0, e_2\}, \{a_1, c_0, f_2\}, \{a_1, d_0, g_2\}, \{b_0, c_0, d_0\}, \{e_2, f_2, g_2\}\}.$$

Then $e = b\beta + a\gamma$, $f = c\beta + a\gamma$, $g = d\beta + a\gamma$ and $\{b\beta, c\beta, d\beta\} = \{e - a\gamma, f - a\gamma, g - a\gamma\} \in \mathcal{B}$; hence by Lemma 9.2.2 $\chi(\beta) = 1$, a contradiction.

If the apex has level 2, we can assume that

$$\mathcal{D} = \{\{a_2, b_0, e_1\}, \{a_2, c_0, f_1\}, \{a_2, d_0, g_1\}, \{b_0, c_0, d_0\}, \{e_1, f_1, g_1\}\}.$$

Then $a = b\beta + e\gamma = c\beta + f\gamma = d\beta + g\gamma$, $\{-b\beta/\gamma, -c\beta/\gamma, -d\beta/\gamma\} = \{e - a/\gamma, f - a/\gamma, g - a/\gamma\} \in \mathcal{B}$, and, again by Lemma 9.2.2, $\chi(-\beta/\gamma) = 1$, a contradiction.

**Case (ii.b) One horizontal block contains the apex.**

The two disjoint blocks must be vertical, and there are three subcases to consider.

If the horizontal block has level 0, we can assume that

$$\mathcal{D} = \{\{a_0, b_0, e_0\}, \{a_0, c_1, f_2\}, \{a_0, d_2, g_1\}, \{b_0, c_1, d_2\}, \{e_0, f_2, g_1\}\}.$$

Then $d = a\beta + g\gamma = b\beta + c\gamma$ and $f = a\beta + c\gamma = e\beta + g\gamma$.

If the horizontal block has level 1, we can assume that

$$\mathcal{D} = \{\{a_1, b_1, e_1\}, \{a_1, c_0, f_2\}, \{a_1, d_2, g_0\}, \{b_1, c_0, d_2\}, \{e_1, f_2, g_0\}\}.$$

Then $d = g\beta + a\gamma = c\beta + b\gamma$ and $f = c\beta + a\gamma = g\beta + e\gamma$.

If the horizontal block has level 2, we can assume that

$$\mathcal{D} = \{\{a_2, b_2, e_2\}, \{a_2, c_0, f_1\}, \{a_2, d_1, g_0\}, \{b_2, c_0, d_1\}, \{e_2, f_1, g_0\}\}.$$

Then $a = c\beta + f\gamma = g\beta + d\gamma$, $b = c\beta + d\gamma$ and $e = g\beta + f\gamma$.

In each of these three subcases we have $a - b = e - a$, a contradiction, since, by transitivity, the block $\{a, b, e\}$ of $S$ cannot have identical differences $a-b$ and $e-a$.

**Case (iii) The crown.**

Let $\{\{a', b', c'\}, \{a', d', e'\}, \{b', d', f'\}, \{c', d', g'\}, \{b', e', h'\}, \{f', g', h'\}\}$ be a crown in $S'$. It is easy to see that $\{c', d', g'\}$ and $\{b', e', h'\}$ must be horizontal blocks at different levels and that all other blocks must be vertical. There are six possible combinations of the levels of these horizontal blocks, but consideration may be reduced to three by noting that $\pi = (b'\ d')(c'\ e')(g'\ h')$ is an automorphism of the crown which exchanges $\{c', d', g'\}$ and $\{b', e', h'\}$.

If the horizontal blocks are $\{c_0, d_0, g_0\}$ and $\{b_1, e_1, h_1\}$ (corresponding to $\{c', d', g'\}$ and $\{b', e', h'\}$, respectively) and the other points are $a_2$ and $f_2$ (corresponding to $a'$ and $f'$), then $a = c\beta + b\gamma = d\beta + e\gamma$ and $f = g\beta + h\gamma = d\beta + b\gamma$. Hence $b - e = (d - c)\beta/\gamma$ and $h - b = (d - g)\beta/\gamma$.

If the horizontal blocks are $\{c_0, d_0, g_0\}$ and $\{b_2, e_2, h_2\}$ and the other points are $a_1$ and $f_1$, then $b = c\beta + a\gamma = d\beta + f\gamma$, $e = d\beta + a\gamma$ and $h = g\beta + f\gamma$. Hence $b - e = -(d - c)\beta$ and $h - b = -(d - g)\beta$.

If the horizontal blocks are $\{c_1, d_1, g_1\}$ and $\{b_2, e_2, h_2\}$ and the other points are $a_0$ and $f_0$, then $b = a\beta + c\gamma = f\beta + d\gamma$, $e = a\beta + d\gamma$ and $h = f\beta + g\gamma$. Hence $b - e = -(d - c)\gamma$ and $h - b = -(d - g)\gamma$.

In each of these three cases we obtain a contradiction by Lemma 9.2.3.

### Case (iv) The 6-cycle.

Let $\{\{a', b', c'\}, \{a', d', e'\}, \{b', d', f'\}, \{c', f', h'\}, \{e', f', g'\}, \{a', g', h'\}\}$ be a 6-cycle in $S'$. It is straightforward to show that either there are precisely two horizontal blocks at different levels, or all blocks are vertical.

### Case (iv.a) Two horizontal blocks at different levels.

By symmetry we may assume that the horizontal blocks are $\{a', g', h'\}$ and $\{b', d', f'\}$. There are six possible combinations for the two levels involved but, again by symmetry, consideration can be reduced to three.

If the horizontal blocks are $\{a_0, g_0, h_0\}$ and $\{b_1, d_1, f_1\}$, let the other points be $c_2$ and $e_2$. Then $c = a\beta + b\gamma = h\beta + f\gamma$ and $e = a\beta + d\gamma = g\beta + f\gamma$. Hence $b - d = (h - g)\beta/\gamma$ and $b - f = (h - a)\beta/\gamma$.

If the horizontal blocks are $\{a_0, g_0, h_0\}$ and $\{b_2, d_2, f_2\}$, let the other points be $c_1$ and $e_1$. Then $b = a\beta + c\gamma$, $d = a\beta + e\gamma$ and $f = h\beta + c\gamma = g\beta + e\gamma$. Hence $b - d = -(h - g)\beta$ and $b - f = -(h - a)\beta$.

If the horizontal blocks are $\{a_1, g_1, h_1\}$ and $\{b_2, d_2, f_2\}$, let the other points be $c_0$ and $e_0$. Then $b = c\beta + a\gamma$, $d = e\beta + a\gamma$ and $f = c\beta + h\gamma = e\beta + g\gamma$. Hence $b - d = -(h - g)\gamma$ and $b - f = -(h - a)\gamma$.

In each of these three cases we obtain a contradiction by Lemma 9.2.4, since none of $\chi(\beta/\gamma), \chi(-\beta), \chi(-\gamma)$ takes the value 1.

**Case (iv.b) There are no horizontal blocks.**

It is easy to show that the two points of degree 3, $a'$ and $f'$, are at the same level. There are then three possibilities.

If the points of degree 3 have level 0, we may assume that the 6-cycle is

$$\{\{a_0, b_2, c_1\}, \{a_0, d_1, e_2\}, \{b_2, d_1, f_0\}, \{c_1, f_0, h_2\}, \{e_2, f_0, g_1\}, \{a_0, g_1, h_2\}\}.$$

Then $b = a\beta + c\gamma = f\beta + d\gamma$, $e = a\beta + d\gamma = f\beta + g\gamma$ and $h = a\beta + g\gamma = f\beta + c\gamma$. Hence $c = d = g$, a contradiction since $c_1$, $d_1$ and $g_1$ are at the same level.

If the points of degree 3 have level 1, we may assume that the 6-cycle is

$$\{\{a_1, b_2, c_0\}, \{a_1, d_0, e_2\}, \{b_2, d_0, f_1\}, \{c_0, f_1, h_2\}, \{e_2, f_1, g_0\}, \{a_1, g_0, h_2\}\}.$$

Then $b = c\beta + a\gamma = d\beta + f\gamma$, $e = d\beta + a\gamma = g\beta + f\gamma$, $h = g\beta + a\gamma = c\beta + f\gamma$ and, again, $c = d = g$, a contradiction.

If the points of degree 3 have level 2, we may assume that the 6-cycle is

$$\{\{a_2, b_1, c_0\}, \{a_2, d_0, e_1\}, \{b_1, d_0, f_2\}, \{c_0, f_2, h_1\}, \{e_1, f_2, g_0\}, \{a_2, g_0, h_1\}\}.$$

Then $a = c\beta + b\gamma = d\beta + e\gamma = g\beta + h\gamma$, $f = d\beta + b\gamma = c\beta + h\gamma = g\beta + e\gamma$. Hence $b = e = h$, a contradiction, and this completes the proof. □

It is worth remarking that with minor changes the argument employed in case (iv.b) works for any $k$-cycle configuration where $k \geq 6$ is even. Moreover, if $\mathcal{C}$ is a $k$-cycle and $k \not\equiv 0 \pmod 6$, it is not possible to assign levels 0, 1 and 2 to the points of $\mathcal{C}$ in the manner described above unless either all blocks of $\mathcal{C}$ are horizontal at the same level, or all blocks of $\mathcal{C}$ are vertical. Hence, recalling that the Pasch configuration is a 4-cycle, we have the following extension of Theorem 9.3.1: *For $k \not\equiv 0 \pmod 6$, if $S$ has no $k$-cycles then $S'$ also has no $k$-cycles.*

The next theorem extends Theorem 9.3.1 to a general product construction.

**Theorem 9.3.2** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$, and $V = GF(v)$. Suppose also that $S^* = (W, \mathcal{B}^*)$ is a Steiner triple system of order $w$. For each block of $\mathcal{B}^*$ arbitrarily fix the order of the points, so that $\mathcal{B}^*$ may be regarded as a set of ordered triples $(i, j, k)$ . Put*

*$V' = V \times W$ and let*

$$\begin{aligned} \mathcal{B}' &= \{\{a_i, b_i, c_i\} : \{a,b,c\} \in \mathcal{B}, i \in W\} \\ &\cup \ \{\{x_i, y_j, (x\beta + y\gamma)_k\} : x, y \in GF(v), \ (i,j,k) \in \mathcal{B}^*\}, \end{aligned}$$

*where $\beta, \gamma \neq 0$ are fixed parameters in $GF(v)$. Then $S' = (V', \mathcal{B}')$ is a Steiner triple system of order $vw$. Furthermore*

*(i) if both $S$ and $S^*$ are anti-Pasch, then $S'$ is also anti-Pasch;*

*(ii) if both $S$ and $S^*$ are anti-mitre, $\alpha^2 \neq \alpha - 1$, and $\chi(\beta), \chi(\gamma), \chi(-\beta/\gamma) \neq 1$, then $S'$ is also anti-mitre;*

*(iii) if both $S$ and $S^*$ have no crowns and $\alpha^2 \notin \{1-\alpha, \alpha+1, 3\alpha-1\}$, then $S'$ also has no crowns;*

*(iv) if both $S$ and $S^*$ have no 6-cycles, $\alpha^2 \neq \alpha - 1$, and $\chi(\beta), \chi(\gamma), \chi(-\beta/\gamma) \neq -1$, then $S'$ also has no 6-cycles.*

*As a consequence, if both $S$ and $S^*$ are 6-sparse,*

$$\alpha^2 \notin \{\alpha - 1, 1 - \alpha, \alpha + 1, 3\alpha - 1\}, \tag{9.5}$$

*and $\chi(\beta), \chi(\gamma), \chi(\beta/\gamma) \neq \pm 1$, then $S'$ is also 6-sparse.*

**Proof.** As in the previous theorem, the construction itself is an application of the generalized direct product, and so $S'$ is an STS$(vw)$. If $x_i$ is a point of $S'$, $x \in V, i \in W$, we refer to $i$ as the *level* of $x_i$. As before, a block of $S'$ is *horizontal* if all of its points have the same level; otherwise it is *vertical.* The elements of a vertical block have distinct levels which, as points of $W$, form a block of $S^*$.

Suppose that $\mathcal{C}$ is one of the configurations in question (Pasch, mitre, crown or 6-cycle) and that $\mathcal{C}$ is present in $S'$ but not in $S$ and $S^*$. Let

$$\mathcal{C}^* = \{\{i,j,k\} : \{x_i, y_j, z_k\} \in \mathcal{C}, \ i \neq j\}.$$

Clearly, $S^*$ contains $\mathcal{C}^*$ and therefore if $\mathcal{C}^* \cong \mathcal{C}$, we have a contradiction. Also, if $\mathcal{C}^*$ is a single block, we can relabel it as $\{0,1,2\}$ and then the proof of this theorem proceeds exactly as in Theorem 9.3.1. We now establish that these are the only possibilities for each of the four configurations: Pasch, mitre, crown and 6-cycle.

**Case (i) The Pasch configuration.**

As in Theorem 9.3.1 we can assume that all blocks of $\mathcal{C}$ are vertical. Then it is easy to see that either $|\mathcal{C}^*| = 1$ or $\mathcal{C}^* \cong \mathcal{C}$.

**Case (ii) The mitre.**

Either the two parallel blocks of the mitre are horizontal, or there is precisely one horizontal block, which contains the point of degree 3, or all blocks are vertical. In the first two cases $|\mathcal{C}^*| = 1$ and in the third case $\mathcal{C}^* \cong \mathcal{C}$.

**Case (iii) The crown.**

Let $\{\{a', b', c'\}, \{a', d', e'\}, \{b', d', f'\}, \{c', d', g'\}, \{b', e', h'\}, \{f', g', h'\}\}$ be a crown in $S'$. We can assume that either $\{c', d', g'\}$ and $\{b', e', h'\}$ are horizontal blocks at different levels and all other blocks are vertical, or all six blocks are vertical. In the former case $|\mathcal{C}^*| = 1$; in the latter case $\mathcal{C}^* \cong \mathcal{C}$.

**Case (iv) The 6-cycle.**

Either there are precisely two horizontal blocks at different levels, or all blocks are vertical. In the former case $|\mathcal{C}^*| = 1$; in the latter case either $|\mathcal{C}^*| = 1$ or $\mathcal{C}^* \cong \mathcal{C}$. This completes the proof. $\square$

By applying the previous two theorems to the 6-sparse systems identified in section 9.2, we can prove the following.

**Theorem 9.3.3** *There are infinitely many 6-sparse Steiner triple systems.*

**Proof.** It is easily verified that property (9.5) holds for each of the systems listed in Table 2. Therefore we can repeatedly apply Theorem 9.3.2, choosing, for example, $\beta = \alpha$ and $\gamma = 1/\alpha$. $\square$

## 9.4 Limitations of the basic construction

In this section we show that the basic construction of Theorem 9.2.1 can never generate more than a finite number of 6-sparse Steiner triple systems. Indeed, we prove that Table 9.2 is complete by showing that the construction produces no 6-sparse STS($v$)s with $v > 9\,150\,625$. Specifically, we prove the existence of a mitre in all such systems where $\alpha^2 - \alpha + 1 \neq 0$. In order to establish this result, we make use of Theorem 9.4.1, below, which is a consequence of a result of André Weil. Our method is similar to the technique used by Ian Anderson and Leigh Ellison to establish the existence of certain triplewhist tournament designs [2, 3].

**Theorem 9.4.1** *Let $\chi$ be a multiplicative character of order $m > 1$ of $GF(q)$, and suppose that the polynomial $f(x)$ over $GF(q)$ has $d$ distinct zeros in the algebraic closure of $GF(q)$ and is not a constant multiple of an $m^{th}$ power. Then*

$$\left| \sum_{x \in GF(q)} \chi(f(x)) \right| \leq (d-1)q^{1/2}.$$

**Proof.** See [66], page 43. □

We now use this result to establish a useful lemma.

**Lemma 9.4.1** *Suppose that $v$ is prime and that $\chi$ is a multiplicative character of $GF(v)$ of order 6. Suppose also that $f_1(x), f_2(x), \ldots, f_n(x)$ are polynomials over $GF(v)$ of degree 1 in $x$, having distinct roots $\rho_1, \rho_2, \ldots, \rho_n$ respectively, with the additional property that for each $i$ $(1 \leq i \leq n)$ there exists $j$ $(1 \leq j \leq n)$ for which*

*addi* $\chi(f_j(\rho_i)) \neq 1$. *Then if*

$\chi(f_j(\rho_i)) \neq 1$. *Then if*

$$v > \left( \sum_{k=2}^{n} (k-1) \binom{n}{k} 5^k \right)^2 = \left( 6^{n-1}(5n-6) + 1 \right)^2, \tag{9.6}$$

*there exists $x \in GF(v)$ such that*

$$\chi(f_1(x)) = \chi(f_2(x)) = \ldots = \chi(f_n(x)) = 1. \tag{9.7}$$

**Proof.** Observe first that the possible values of $\chi(x)$ are the six sixth roots of unity when $x \neq 0$, and $\chi(0) = 0$. Put

$$\pi(x) = \left( \sum_{i_1=0}^{5} \chi(f_1(x)^{i_1}) \right) \left( \sum_{i_2=0}^{5} \chi(f_2(x)^{i_2}) \right) \cdots \left( \sum_{i_n=0}^{5} \chi(f_n(x)^{i_n}) \right).$$

If (9.7) holds then $\pi(x) = 6^n$, while if $\chi(f_j(x)) \neq 1$ (including the possibility that $x = \rho_j$) then $\pi(x) = 0$. Thus $\pi(x) \neq 0$ if and only if (9.7) holds.

Next put $\Delta = \sum_{x\in GF(v)} \pi(x)$. Note that if we can prove that $\Delta \neq 0$ then it will follow that there exists an $x \in GF(v)$ which satisfies (9.7). But $\pi(x)$ has the form

$$\pi(x) = 1 + \sum_{\substack{i_1,i_2,\ldots,i_n=0 \\ i_1+i_2+\ldots+i_n\neq 0}}^{5} \chi((f_1(x))^{i_1}(f_2(x))^{i_2}\ldots(f_n(x))^{i_n}).$$

So

$$\Delta = v + \sum_{\substack{i_1,i_2,\ldots,i_n=0 \\ i_1+i_2+\ldots+i_n\neq 0}}^{5} \sum_{x\in GF(v)} \chi((f_1(x))^{i_1}(f_2(x))^{i_2}\ldots(f_n(x))^{i_n}).$$

Since the $f_i(x)$ are all first order polynomials in $x$ with distinct roots, a product of the form $(f_1(x))^{i_1}(f_2(x))^{i_2}\ldots(f_n(x))^{i_n}$ with $0 \leq i_1, i_2, \ldots, i_n \leq 5$ cannot be a constant multiple of a sixth power of a polynomial in $x$ unless $i_1 = i_2 = \ldots = i_n = 0$. Hence, by Theorem 9.4.1 and provided that $i_1 + i_2 + \ldots + i_n \neq 0$, we have

$$\left|\sum_{x\in \mathrm{GF}(v)} \chi((f_1(x))^{i_1}(f_2(x))^{i_2}\ldots(f_n(x))^{i_n})\right| \leq (r_{i_1,i_2,\ldots,i_n} - 1)\sqrt{v},$$

where $r_{i_1,i_2,\ldots,i_n}$ is the number of distinct roots of $f_1^{i_1}f_2^{i_2}\ldots f_n^{i_n}$ in $GF(v)$. But $r_{i_1,i_2,\ldots,i_n}$ is precisely the number of non-zero indices amongst $\{i_1, i_2, \ldots, i_n\}$ in the expression $\chi((f_1(x))^{i_1}(f_2(x))^{i_2}\ldots(f_n(x))^{i_n})$. Hence

$$|\Delta| \geq v - \sum_{k=2}^{n}(k-1)\binom{n}{k}5^k\sqrt{v}.$$

This is strictly positive if (9.6) holds. □

With the aid of the preceding lemma we can prove the following theorem.

**Theorem 9.4.2** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. Then if $v > 9\,150\,625$ and $\alpha^2 - \alpha + 1 \neq 0$, $S$ contains a mitre.*

**Proof.** Consider the following five sets of points.

$$\begin{gathered}
\{0, 1, \alpha\}, \\
\{0, x, \alpha x\} = x\{0, 1, \alpha\}, \\
\{\alpha x, 1, \alpha + \alpha x - \alpha^2 x\} = (1 - \alpha x)\{0, 1, \alpha\} + \alpha x, \\
\left\{\frac{\alpha(1-\alpha x)}{1-\alpha}, \alpha x, \alpha\right\} = \left(\frac{\alpha(x-1)}{1-\alpha}\right)\{0, 1, \alpha\} + \frac{\alpha(1-\alpha x)}{1-\alpha}, \\
\left\{\frac{\alpha(1-\alpha x)}{1-\alpha}, x, \alpha + \alpha x - \alpha^2 x\right\} = \left(\frac{(\alpha^2-\alpha+1)x-\alpha}{1-\alpha}\right)\{0, 1, \alpha\} + \frac{\alpha(1-\alpha x)}{1-\alpha}.
\end{gathered}$$

These are five distinct blocks of $S$ provided that $x$ is selected to satisfy the following relationships.

$$\chi(x)=1,\ \chi(1-\alpha x)=1,\ \chi\left(\frac{\alpha(x-1)}{1-\alpha}\right)=1,\ \chi\left(\frac{(\alpha^2-\alpha+1)x-\alpha}{1-\alpha}\right)=1.$$

So put $f_1(x)=x,\ f_2(x)=1-\alpha x,\ f_3(x)=\frac{\alpha(x-1)}{1-\alpha},\ f_4(x)=\frac{(\alpha^2-\alpha+1)x-\alpha}{1-\alpha}$. Then, provided that $\alpha^2-\alpha+1\neq 0$, each $f_i(x)$ is a polynomial of degree 1 in $x$. These four polynomials have the distinct roots $\rho_1=0,\ \rho_2=\frac{1}{\alpha},\ \rho_3=1,\ \rho_4=\frac{\alpha}{\alpha^2-\alpha+1}$. It is also easy to verify that for each $i$ there exists $j$ with $\chi(f_j(\rho_i))\neq 1$. For example, $f_3(\rho_4)=(\alpha-1)f_1(\rho_4)$, so either for $j=1$ or for $j=3$ we have $\chi(f_j(\rho_4))\neq 1$. By applying the previous lemma, we find that there exists a suitable $x\in \mathrm{GF}(v)$ provided that $v>(6^3\cdot 14+1)^2=3025^2=9\,150\,625$. But then the five blocks form a mitre in $S$ and the proof is complete. □

**Theorem 9.4.3** *Suppose that $S=(V,\mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. Then if $v>9\,150\,625$, $S$ is not 6-sparse.*

**Proof.** In view of Theorem 9.4.2 it is only necessary to consider the case when $\alpha^2-\alpha+1=0$. Then $\alpha$ is a primitive sixth root of unity (which entails $v\not\equiv 19 \pmod{36}$) and the system $S$ is the so-called Netto system described in [61, 16]. It is shown in [61] that such systems contain Pasch configurations when $v\equiv 7 \pmod{24}$, and in [35], using a result from [61], it is shown that such systems contain 6-cycles when $v\equiv 19 \pmod{24}$. □

The method by which the mitre of Theorem 9.4.2 was found can be used to search for other configurations. Let $S$ be a block transitive STS($v$) with parameter $\alpha$, and let $\mathcal{C}$ be a configuration of $n$ blocks in $S$. By block transitivity, we can assume that one of the blocks of $\mathcal{C}$ is $\{0,1,\alpha\}$. Denoting the other $n-1$ blocks of $\mathcal{C}$ by $\{x_i,y_i,z_i\}$, $i=1,2,\ldots,n-1$, we can set up $6^{n-1}$ sets of simultaneous equations

$$(x_i,y_i,z_i)=\pi_i((0,1,\alpha))\mu_i+m_i,\ i=1,2,\ldots,n-1, \tag{9.8}$$

where $\alpha$ is considered to be a constant and $\pi_i\in S_3$ is a permutation of three elements. The points $x_1$, $y_1$, $z_1$, $x_2$, $y_2$, $z_2$, $\ldots x_{n-1}$, $y_{n-1}$, $z_{n-1}$ are not necessarily distinct from each other nor from 0, 1 and $\alpha$.

Attempting to solve these sets of equations, one makes the following observations. Suppose $\mathcal{C}$ has $n$ blocks and at least $n+3$ points. There are $3n-3$ equations and at least $3n-2$ variables, namely $n-1$ $\mu_i$ variables, $n-1$ $m_i$ variables and at least $n$ point variables (points 0, 1 and $\alpha$ are not counted). So there is for most choices of the $\pi_i$ a solution with at least one free parameter. Hence we expect these configurations to be unavoidable in block transitive STS($v$)s for sufficiently large $v$. On the other hand, if $\mathcal{C}$ has $n$ blocks and $n+2$ points, then in most cases where there is a solution to (9.8) it will not have a free parameter. However, exceptions can occur; it is possible that there might be instances where there is some redundancy in the equations, resulting in a solution set of positive degree. Indeed, it turns out that this is the case for the mitre.

If we consider a mitre of the form

$$\{\{0,c,x\},\{1,c,y\},\{\alpha,c,z\},\{x,y,z\},\{0,1,\alpha\}\}, \tag{9.9}$$

there are indeed precisely two sets of equations,

$$\begin{aligned}(0,c,x) &= (0,\alpha,1)\mu_1 + m_1,\\ (1,c,y) &= (1,0,\alpha)\mu_2 + m_2,\\ (\alpha,c,z) &= (\alpha,1,0)\mu_3 + m_3,\\ (x,y,z) &= (1,\alpha,0)\mu_4 + m_4\end{aligned} \tag{9.10}$$

and

$$\begin{aligned}(0,c,x) &= (0,1,\alpha)\mu_1 + m_1,\\ (1,c,y) &= (1,\alpha,0)\mu_2 + m_2,\\ (\alpha,c,z) &= (\alpha,0,1)\mu_3 + m_3,\\ (x,y,z) &= (\alpha,0,1)\mu_4 + m_4,\end{aligned} \tag{9.11}$$

which have a free parameter in their solution. Designating $\mu = \mu_1$ as the free parameter, we therefore have two possible mitres (9.9) for any specific $\mu$,

$$\begin{aligned}&c=\alpha\mu,\ x=\mu,\ y=\alpha+\alpha\mu-\alpha^2\mu,\ z=\frac{\alpha(1-\alpha\mu)}{1-\alpha},\\ &\mu_2 = 1-\alpha\mu,\ \mu_3=\frac{\alpha(\mu-1)}{1-\alpha},\ \mu_4=\frac{\alpha^2\mu-\alpha\mu+\mu-\alpha}{1-\alpha},\\ &m_1=0,\ m_2=c,\ m_3=m_4=z\end{aligned} \tag{9.12}$$

and

$$\begin{aligned}
&c=\mu,\ \ x=\alpha\mu,\ y=\frac{\mu-\alpha}{1-\alpha},\ z=1+\mu-\frac{\mu}{\alpha},\\
&\mu_2=\frac{1-\mu}{1-\alpha},\ \mu_3=1-\frac{\mu}{\alpha},\ \mu_4=\mu+\frac{\alpha-\mu}{\alpha-\alpha^2},\\
&m_1=0,\ m_2=m_4=y,\ m_3=c.
\end{aligned}\tag{9.13}$$

It is easily seen that under the mapping $\alpha\mapsto(1/\alpha)$, the multipliers $\mu_2$, $\mu_3$ and $\mu_4$ in (9.12) are transformed to $\mu_3$, $\mu_2$ and $\mu_4$, respectively, in (9.13). Therefore, since the STS($v$) with parameter $\alpha$ is isomorphic to the STS($v$) with parameter $1/\alpha$, if one of the mitres defined by (9.12) and (9.13) exists in $S$ for a specific value of $\mu$, then so does the other. Moreover, they are distinct because they have different apexes. The first mitre, (9.12) is the one that features in the proof of Theorem 9.4.2.

Another configuration to which we can apply this technique is $F_{13}$, the 7-block, 9-point configuration that can be obtained by adding a diagonal to the 'window-frame' configuration $E_5$.

$$F_{13}:\{012,034,135,246,257,168,078\},\quad |\mathrm{Aut(F_{13})}|=12.$$

In fact, we encounter many parametric solutions of (9.8) and it would take us too long to examine them all. Here we mention just one, which is realized as

$$\begin{aligned}
\mathcal{F}\ =\ &\{\{1-\mu+\alpha\mu,\ \alpha\mu,\alpha\},\\
&\{1-\mu,0,\alpha(1-\mu)\},\ \{1,\mu,\mu+\alpha(1-\mu)\},\\
&\{1-\mu+\alpha\mu,1-\alpha,1\},\ \{\alpha\mu,0,\mu\},\\
&\{\alpha,\alpha(1-\mu),\ \mu+\alpha(1-\mu)\},\ \{0,1,\alpha\}\},
\end{aligned}$$

provided $\mu$ can be chosen such that $\chi(\mu)\equiv\chi(1-\mu)=1$ and

$$\mu\notin\left\{0,1,\frac{1}{2},\frac{1}{1+\alpha},\frac{1-\alpha}{1-2\alpha},\frac{\alpha}{2\alpha-1},\frac{1-\alpha}{2-\alpha},\frac{1}{2-\alpha},\frac{\alpha}{1+\alpha}\right\},$$

the latter condition ensuring that $\mathcal{F}$ does indeed have nine distinct points. As in Theorem 9.4.2, we can show that if $v$ is sufficiently large, for each STS($v$) arising from Theorem 9.2 we can find a suitable $\mu$, thereby proving that the system contains an $F_{13}$.

## 9.5 Further 6-sparse Steiner triple systems

In this section we give a construction analogous to Theorem 9.2.1 for the case $v \equiv 9 \pmod{12}$ and we show that it produces 6-sparse systems for arbitrarily large $v$.

**Theorem 9.5.1** *Let $p = 2s + 1 \geq 7$ be a prime such that $p \equiv 3 \pmod 4$ and let $v = 3p$. Let $\tau$ be an integer modulo $v$ such that $\tau \not\equiv 0 \pmod 3$ and $\tau$ is a primitive root modulo $p$. Let $\omega = \tau^2 \bmod v$. Choose $\alpha$ modulo $v$ such that either (i) $\alpha \equiv 0 \pmod 3$ and $((\alpha - 1)/p) = 1$, or (ii) $\alpha \equiv 1 \pmod 3$ and $(-\alpha/p) = 1$. Then, with all arithmetic modulo $v$,*

$$\{\{m, m + \omega^i, m + \alpha\omega^i\} : i = 0, 1, \ldots, s - 1, m = 0, 1, \ldots, v - 1\}$$
$$\cup \ \{\{n, n + p, n + 2p\} : n = 0, 1, \ldots, p - 1\}$$

*is the set of blocks of an STS($v$), defined on $\{0, 1, \ldots, v - 1\}$, which is generated by $\{0, 1, \alpha\}$ and $\{0, p, 2p\}$ under the action of the group of mappings*

$$G \ = \ \{x \mapsto \omega^i x + m \bmod v, \ i = 0, 1, \ldots, s - 1, \ m = 0, 1, \ldots, v - 1\}.$$

**Proof.** In this proof and the remarks which follow we shall tacitly assume that unless otherwise specified all arithmetic is performed modulo $v$.

Clearly, the orbit of the starter block $\{0, p, 2p\}$ under the action of $G$ is $\{\{n, n + p, n + 2p\} : n = 0, 1, \ldots, p - 1\}$. Let

$$\Omega(x) = \{x\,\omega^i \bmod v : i = 0, 1, \ldots, s - 1\}$$

and observe that for any $x$ modulo $v$, we have

$$\left(\frac{x\omega}{p}\right) = \left(\frac{x}{p}\right), \qquad x\omega \equiv x \pmod 3$$

and

$$\Omega(x) = \left\{y \bmod v : \left(\frac{y}{p}\right) = \left(\frac{x}{p}\right) \text{ and } y \equiv x \pmod 3\right\}.$$

Therefore we can prove the theorem by showing that the six differences $\pm 1$, $\pm\alpha$ and $\pm(1 - \alpha)$ generated by the triple $\{0, 1, \alpha\}$ have distinct combinations of quadratic character modulo $p$ and residue class modulo 3. Since $(-1/p) = -1$, this is possible if and only if $\alpha$ satisfies (i) or (ii) in the statement of the theorem. □

The choice of $\tau$ is immaterial, subject to $\tau \not\equiv 0 \pmod 3$ and $\tau$ being a primitive root modulo $p$. To see this, suppose $\tau' \not\equiv 0 \pmod 3$ is also a primitive root modulo $p$ and let $\omega' = (\tau')^2$. Then $\tau' \equiv \tau^t \pmod p$ for some $t$ with $(t, p-1) = 1$ and it is plain that for any $x$,

$$\Omega(x) = \{x\,(\omega')^i \bmod v : i = 0, 1, \ldots, s-1\}.$$

If $\alpha \equiv 0 \pmod 3$, the four STS($v$)s generated by the blocks $\{0, 1, \delta\}$ and $\{0, v/3, 2v/3\}$ for $\delta \in \{\alpha, 1-\alpha, 1/(1-\alpha), 1-1/(1-\alpha)\}$ are isomorphic since the orbits of the generating blocks are invariant under the mappings $x \mapsto 1-x$, $x \mapsto (x-1)/(\alpha-1)$ and $x \mapsto (\alpha-x)/(\alpha-1)$.

If $\alpha \equiv 1 \pmod 3$, the four STS($v$)s generated by the blocks $\{0, 1, \delta\}$ and $\{0, v/3, 2v/3\}$ for $\delta \in \{\alpha, 1-\alpha, 1/\alpha, 1-1/\alpha\}$ are isomorphic since the orbits of the generating blocks are invariant under the mappings $x \mapsto 1-x$, $x \mapsto x/\alpha$ and $x \mapsto 1-x/\alpha$.

A complete list of 6-sparse Steiner triple systems created by Theorem 9.5.1 for $v < 10000$ is given in Table 9.5. Systems with the same value of $v$ are pairwise non-isomorphic, as can be seen by examining the structures of the cycle graphs $G_{0,1}$, $G_{0,\alpha}$, $G_{1,\alpha}$ and $G_{0,v/3}$. (See section 10.1 for the definition of $G_{x,y}$.) With this list we can then use Theorem 9.3.2 to prove the existence of many new orders of 6-sparse Steiner triple systems, provided that the STS($v$) in that theorem is one of the systems with $v \equiv 7 \pmod{12}$ given by Theorem 9.2.1.

There is no blocking mechanism to prevent the formation of 6-sparse systems. Indeed, the two special mitres that become unavoidable in the systems of Theorem 9.2.1 do not form in the two-generator systems of Theorem 9.5.1. Thus we expect Theorem 9.5.1 to produce 6-sparse systems with arbitrarily large $v$. We state this as our last theorem but we can only offer a proof which relies on a considerable element of computation.

**Theorem 9.5.2** *For all sufficiently large $v$ with $v = 3p$, $p$ prime and $p \equiv 3 \pmod 4$, there exists $\alpha$ such that the system of Theorem 9.5.1 generated by $\{0, 1, \alpha\}$ and $\{0, p, 2p\}$ is 6-sparse.*

The proof of this theorem makes use of the following lemmas.

Table 9.5: 6-sparse systems with $v \equiv 9 \pmod{12}$

| $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ | $v$ | $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 489 | 135 | 3837 | 880 | 5277 | 1377 | 6429 | 129 | 7977 | 1960 | 8637 | 919 | 9357 | 18 |
| 501 | 160 | 3849 | 1263 | 5277 | 1486 | 6429 | 1462 | 7977 | 2404 | 8637 | 1393 | 9357 | 390 |
| 1077 | 75 | 3909 | 544 | 5277 | 2074 | 6429 | 2097 | 7977 | 2944 | 8637 | 2046 | 9357 | 403 |
| 1101 | 379 | 3909 | 1063 | 5349 | 15 | 6537 | 915 | 7989 | 402 | 8637 | 2077 | 9357 | 1033 |
| 1149 | 328 | 3981 | 1627 | 5361 | 835 | 6537 | 1068 | 7989 | 657 | 8637 | 4141 | 9357 | 1516 |
| 1329 | 309 | 4101 | 265 | 5361 | 1075 | 6609 | 31 | 7989 | 2298 | 8661 | 490 | 9357 | 2152 |
| 1437 | 12 | 4101 | 427 | 5361 | 1377 | 6609 | 810 | 7989 | 3429 | 8661 | 1011 | 9357 | 2403 |
| 1461 | 13 | 4101 | 561 | 5469 | 84 | 6717 | 954 | 8013 | 348 | 8661 | 1254 | 9357 | 2643 |
| 1461 | 42 | 4281 | 204 | 5469 | 415 | 6753 | 1551 | 8013 | 496 | 8661 | 1918 | 9489 | 1048 |
| 1509 | 490 | 4317 | 201 | 5469 | 1114 | 6753 | 2184 | 8013 | 549 | 8661 | 2901 | 9489 | 1191 |
| 1569 | 232 | 4317 | 432 | 5469 | 1516 | 6861 | 385 | 8013 | 793 | 8709 | 42 | 9489 | 1809 |
| 1641 | 223 | 4317 | 658 | 5493 | 430 | 6861 | 604 | 8013 | 1009 | 8709 | 99 | 9489 | 4314 |
| 1689 | 276 | 4317 | 693 | 5493 | 1576 | 6933 | 933 | 8013 | 2353 | 8709 | 250 | 9501 | 168 |
| 1857 | 141 | 4317 | 744 | 5541 | 1104 | 6933 | 3030 | 8049 | 570 | 8709 | 705 | 9501 | 471 |
| 1857 | 328 | 4317 | 993 | 5541 | 1707 | 7017 | 81 | 8049 | 1173 | 8709 | 1296 | 9501 | 486 |
| 1929 | 502 | 4353 | 660 | 5541 | 2344 | 7017 | 240 | 8061 | 3 | 8709 | 1395 | 9501 | 1605 |
| 1929 | 508 | 4353 | 1057 | 5601 | 1065 | 7017 | 1117 | 8061 | 18 | 8709 | 1695 | 9501 | 2514 |
| 1941 | 3 | 4377 | 58 | 5613 | 1470 | 7041 | 351 | 8061 | 57 | 8709 | 2010 | 9501 | 3609 |
| 1941 | 736 | 4377 | 184 | 5613 | 1900 | 7041 | 1009 | 8061 | 439 | 8709 | 3925 | 9561 | 148 |
| 1977 | 519 | 4377 | 409 | 5613 | 2218 | 7041 | 1305 | 8061 | 576 | 8781 | 366 | 9561 | 4164 |
| 2157 | 36 | 4449 | 94 | 5613 | 2343 | 7041 | 1392 | 8061 | 1270 | 8781 | 498 | 9561 | 4273 |
| 2157 | 186 | 4497 | 430 | 5637 | 880 | 7053 | 520 | 8061 | 1333 | 8781 | 685 | 9573 | 54 |
| 2181 | 9 | 4569 | 370 | 5721 | 1594 | 7053 | 985 | 8061 | 1531 | 8781 | 979 | 9573 | 162 |
| 2217 | 193 | 4569 | 1402 | 5853 | 376 | 7053 | 1650 | 8097 | 666 | 8781 | 2251 | 9573 | 391 |
| 2229 | 880 | 4569 | 1837 | 5853 | 435 | 7053 | 2227 | 8121 | 307 | 8781 | 3706 | 9573 | 687 |
| 2361 | 979 | 4593 | 117 | 5853 | 1677 | 7113 | 2404 | 8121 | 1231 | 8817 | 571 | 9573 | 1093 |
| 2433 | 594 | 4593 | 1210 | 5853 | 2064 | 7149 | 714 | 8121 | 1347 | 8817 | 1552 | 9573 | 1350 |
| 2589 | 684 | 4629 | 366 | 5937 | 1365 | 7197 | 966 | 8133 | 292 | 8817 | 1969 | 9573 | 2085 |
| 2649 | 421 | 4629 | 1699 | 5937 | 1606 | 7197 | 1138 | 8133 | 1386 | 8817 | 2991 | 9573 | 2202 |
| 2649 | 609 | 4677 | 12 | 5961 | 358 | 7233 | 1794 | 8133 | 2764 | 8913 | 694 | 9609 | 306 |
| 2721 | 534 | 4677 | 78 | 5961 | 1540 | 7269 | 85 | 8133 | 3225 | 8913 | 1725 | 9609 | 721 |
| 2733 | 24 | 4677 | 99 | 5997 | 643 | 7341 | 1390 | 8157 | 2062 | 8913 | 3289 | 9609 | 1191 |
| 2733 | 240 | 4677 | 126 | 5997 | 1372 | 7341 | 1597 | 8193 | 160 | 8997 | 150 | 9609 | 1260 |
| 2733 | 585 | 4677 | 583 | 6009 | 360 | 7377 | 891 | 8193 | 1153 | 8997 | 351 | 9609 | 1731 |
| 2733 | 682 | 4677 | 1240 | 6009 | 900 | 7377 | 2287 | 8301 | 700 | 8997 | 367 | 9609 | 1783 |
| 2841 | 447 | 4701 | 76 | 6009 | 1167 | 7401 | 87 | 8301 | 835 | 8997 | 753 | 9609 | 2994 |
| 2949 | 711 | 4701 | 337 | 6033 | 126 | 7401 | 3546 | 8301 | 871 | 8997 | 955 | 9609 | 3166 |
| 2949 | 906 | 4701 | 430 | 6033 | 792 | 7509 | 907 | 8301 | 994 | 8997 | 1227 | 9753 | 2193 |
| 2949 | 919 | 4701 | 499 | 6033 | 2251 | 7509 | 1293 | 8301 | 1011 | 8997 | 2253 | 9753 | 3313 |
| 2973 | 288 | 4749 | 418 | 6081 | 457 | 7509 | 1762 | 8301 | 2398 | 8997 | 2857 | 9753 | 3454 |
| 2973 | 309 | 4749 | 1239 | 6081 | 1360 | 7593 | 103 | 8373 | 537 | 8997 | 3295 | 9777 | 364 |
| 3057 | 954 | 4749 | 1294 | 6117 | 604 | 7593 | 219 | 8373 | 1657 | 8997 | 3606 | 9777 | 903 |
| 3093 | 445 | 4821 | 43 | 6117 | 2373 | 7593 | 1108 | 8373 | 2697 | 9033 | 273 | 9813 | 1743 |
| 3093 | 610 | 4821 | 565 | 6117 | 2490 | 7617 | 85 | 8373 | 2913 | 9033 | 1582 | 9813 | 3049 |
| 3117 | 345 | 4821 | 826 | 6189 | 63 | 7617 | 223 | 8409 | 630 | 9033 | 3421 | 9897 | 1206 |
| 3117 | 579 | 4821 | 1240 | 6189 | 1429 | 7617 | 231 | 8409 | 1927 | 9057 | 397 | 9921 | 96 |
| 3189 | 318 | 4821 | 1587 | 6189 | 2224 | 7617 | 816 | 8409 | 2554 | 9057 | 720 | 9921 | 910 |
| 3261 | 9 | 4857 | 163 | 6249 | 69 | 7617 | 864 | 8457 | 685 | 9057 | 1308 | 9921 | 3514 |
| 3261 | 409 | 4857 | 1057 | 6249 | 561 | 7629 | 141 | 8529 | 57 | 9057 | 2643 | 9921 | 3865 |
| 3261 | 735 | 4881 | 942 | 6249 | 2653 | 7629 | 876 | 8529 | 471 | 9069 | 2761 | 9957 | 99 |
| 3309 | 390 | 4881 | 1761 | 6261 | 907 | 7653 | 162 | 8529 | 507 | 9201 | 486 | 9957 | 144 |
| 3309 | 940 | 4989 | 336 | 6261 | 1200 | 7653 | 366 | 8529 | 3192 | 9201 | 595 | 9957 | 2194 |
| 3453 | 802 | 5001 | 919 | 6261 | 1219 | 7653 | 498 | 8553 | 444 | 9201 | 946 | 9957 | 4138 |
| 3513 | 223 | 5001 | 1530 | 6261 | 1422 | 7653 | 1440 | 8553 | 568 | 9201 | 1327 | 9969 | 619 |
| 3513 | 598 | 5001 | 1608 | 6297 | 286 | 7653 | 1612 | 8553 | 1189 | 9201 | 2146 | 9969 | 2410 |
| 3561 | 313 | 5097 | 70 | 6297 | 1278 | 7737 | 1192 | 8553 | 1738 | 9201 | 2365 | 9969 | 2565 |
| 3669 | 87 | 5097 | 633 | 6297 | 1983 | 7773 | 1327 | 8553 | 2931 | 9237 | 648 | 9993 | 2443 |
| 3669 | 231 | 5097 | 1227 | 6333 | 135 | 7773 | 2185 | 8637 | 52 | 9237 | 693 | | |
| 3669 | 520 | 5097 | 1747 | 6333 | 648 | 7773 | 2239 | 8637 | 232 | 9237 | 2287 | | |
| 3693 | 102 | 5169 | 915 | 6333 | 810 | 7773 | 3270 | 8637 | 432 | 9249 | 556 | | |
| 3693 | 544 | 5241 | 538 | 6333 | 2242 | 7941 | 1864 | 8637 | 523 | 9249 | 3069 | | |
| 3693 | 838 | 5241 | 2160 | 6429 | 72 | 7977 | 1107 | 8637 | 744 | 9249 | 3339 | | |

**Lemma 9.5.1** *Let $n$ be a positive integer, let $p$ be a prime, let*

$$\begin{array}{rcl} a_{1,1}\,x_1 + a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n + c_1 & \equiv & 0 \pmod{p} \\ a_{2,1}\,x_1 + a_{2,2}\,x_2 + \cdots + a_{2,n}\,x_n + c_2 & \equiv & 0 \pmod{p} \\ & \cdots & \\ a_{n,1}\,x_1 + a_{n,2}\,x_2 + \cdots + a_{n,n}\,x_n + c_n & \equiv & 0 \pmod{p} \end{array} \tag{9.14}$$

*be a set of linear congruences modulo $p$ and let $\mathbf{A} = [a_{i,j}]$ be the corresponding matrix of coefficients. Suppose* $\det(\mathbf{A}) \not\equiv 0 \pmod{p}$. *Then there exists a unique solution of (9.14) in GF(p). Furthermore, the solution is the same as that obtained by solving (9.14) over the rationals.*

**Proof.** This is well known. $\square$

**Lemma 9.5.2** *Let $V = \{0, 1, \ldots, v-1\}$ and let $S = (V, \mathcal{B})$ be an STS($v$) of Theorem 9.5.1. Suppose $S$ is not 6-sparse. Let $\Gamma = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{13}\}$, where*

$$\begin{aligned}
\mathcal{G}_1 &= \{\{0,1,\alpha\},\{0,x_1,x_2\},\{1,x_2,x_3\},\{\alpha,x_1,x_3\}\},\\
\mathcal{G}_2 &= \{\{0,1,\alpha\},\{0,x_1,x_2\},\{0,x_3,x_4\},\{1,x_1,x_3\},\{\alpha,x_2,x_4\}\},\\
\mathcal{G}_3 &= \{\{0,1,\alpha\},\{1,x_1,x_2\},\{1,x_3,x_4\},\{0,x_1,x_3\},\{\alpha,x_2,x_4\}\},\\
\mathcal{G}_4 &= \{\{0,1,\alpha\},\{\alpha,x_1,x_2\},\{\alpha,x_3,x_4\},\{0,x_1,x_3\},\{1,x_2,x_4\}\},\\
\mathcal{G}_5 &= \{\{0,1,\alpha\},\{0,x_1,x_2\},\{0,x_3,x_4\},\{x_5,1,x_1\},\{x_5,\alpha,x_3\},\{x_5,x_2,x_4\}\},\\
\mathcal{G}_6 &= \{\{0,1,\alpha\},\{1,x_1,x_2\},\{1,x_3,x_4\},\{x_5,0,x_1\},\{x_5,\alpha,x_3\},\{x_5,x_2,x_4\}\},\\
\mathcal{G}_7 &= \{\{0,1,\alpha\},\{\alpha,x_1,x_2\},\{\alpha,x_3,x_4\},\{x_5,0,x_1\},\{x_5,1,x_3\},\{x_5,x_2,x_4\}\},\\
\mathcal{G}_8 &= \{\{0,1,\alpha\},\{0,x_1,x_2\},\{0,x_3,x_5\},\{1,x_1,x_4\},\{\alpha,x_1,x_5\},\{x_2,x_3,x_4\}\},\\
\mathcal{G}_9 &= \{\{0,1,\alpha\},\{0,x_1,x_2\},\{0,x_3,x_5\},\{\alpha,x_1,x_4\},\{1,x_1,x_5\},\{x_2,x_3,x_4\}\},\\
\mathcal{G}_{10} &= \{\{0,1,\alpha\},\{1,x_1,x_2\},\{1,x_3,x_5\},\{0,x_1,x_4\},\{\alpha,x_1,x_5\},\{x_2,x_3,x_4\}\},\\
\mathcal{G}_{11} &= \{\{0,1,\alpha\},\{1,x_1,x_2\},\{1,x_3,x_5\},\{\alpha,x_1,x_4\},\{0,x_1,x_5\},\{x_2,x_3,x_4\}\},\\
\mathcal{G}_{12} &= \{\{0,1,\alpha\},\{\alpha,x_1,x_2\},\{\alpha,x_3,x_5\},\{0,x_1,x_4\},\{1,x_1,x_5\},\{x_2,x_3,x_4\}\},\\
\mathcal{G}_{13} &= \{\{0,1,\alpha\},\{\alpha,x_1,x_2\},\{\alpha,x_3,x_5\},\{1,x_1,x_4\},\{0,x_1,x_5\},\{x_2,x_3,x_4\}\}.
\end{aligned}$$

*Then there is a $\mathcal{G} \in \Gamma$ such that $\mathcal{G} \subset \mathcal{B}$ for some $x_1, x_2, \ldots, x_n \in V$, where $n = |\mathcal{G}|-1$.*

**Proof.** By Theorem 9.5.1, $v = 3p$, $p$ prime, $p \equiv 3 \pmod 4$, and $S$ is generated by blocks $\{0, 1, \alpha\}$ and $\{0, p, 2p\}$. Let $\mathcal{X}$ be one of the configurations Pasch, mitre, 6-cycle, crown. Suppose $\mathcal{X} \subset \mathcal{B}$.

Observe that $\mathcal{G}_1$ is a Pasch configuration, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are mitres, $\mathcal{G}_5$, $\mathcal{G}_6$ and $\mathcal{G}_7$ are 6-cycles, $\mathcal{G}_8$, $\mathcal{G}_9$, $\ldots$, $\mathcal{G}_{13}$ are crowns and that the block of $\mathcal{G} \in \Gamma$ labelled $\{0, 1, \alpha\}$ is one of two intersecting blocks which map to each other under an automorphism of $\mathcal{G}$. Since $\mathcal{X}$ cannot contain two intersecting blocks belonging to the orbit of $\{0, p, 2p\}$, it is straightforward to verify (perhaps by the drawing of suitable diagrams) that there exists an automorphism of $S$ which maps $\mathcal{X}$ to some $\mathcal{G} \in \Gamma$ for some $x_1, x_2, \ldots, x_{|\mathcal{G}|-1} \in V$. $\square$

**Lemma 9.5.3** *Let $p$ be prime and suppose that the polynomial $f(x)$ is not a constant multiple of a square over $GF(p)$. Then*

$$\left| \sum_{x \in GF(p)} \left( \frac{f(x)}{p} \right) \right| = O(\sqrt{p}).$$

**Proof.** This is a special case of Theorem 9.4.1. $\square$

**Lemma 9.5.4** *Let*

$$\begin{aligned}\Lambda = \{&x, x-1, x+1, -2x+1, 2x-3, -x+3, x^2+1, -x^2-2,\\ &-x^2-x+1, x^2-x+1, -x^2+x+1, -x^2+2x-2,\\ &-x^2+3x-3, -2x^2+3x-2, 3x^2-4x+2, 2x^2-4x+3,\\ &-2x^2+3x-3, 3x^2-5x+3, x^2-2x+3, x^2-3x+1,\\ &-x^3+x^2-1, -x^3+2x^2-x-1, -x^3+3x^2-2x+1,\\ &x^3-2x^2+3x-3, x^3-3x^2+6x-3, x^3-3x+3,\\ &-x^3+5x^2-6x+3, -x^3+3x^2-4x+1\}.\end{aligned}$$

*Given a fixed positive number $N$, for all sufficiently large prime $p$, there exist at least $N$ numbers $\alpha$, distinct modulo $p$, such that $(\lambda(\alpha)/p) = 1$ for all $\lambda(x) \in \Lambda$.*

**Proof.** Suppose not. Let

$$\pi(x) = \prod_{\lambda(x)\in\Lambda} \left(1 + \left(\frac{\lambda(x)}{p}\right)\right) \tag{9.15}$$

and

$$\Delta = \sum_{x\in \mathrm{GF}(p)} \pi(x).$$

Since $\pi(x) = 0$ if $(\lambda(x)/p) = -1$ for some $\lambda(x) \in \Lambda$, there exists an infinite sequence $\mathfrak{P}$ of primes such that the number of $x$ for which $\pi(x) \neq 0$ is bounded as $p \to \infty$ through members of $\mathfrak{P}$. Hence $\Delta = O(1)$ as $p \to \infty$ through members of $\mathfrak{P}$. But

$$\Delta = p + \sum_{f(x)} \sum_{x\in \mathrm{GF}(p)} \left(\frac{f(x)}{p}\right),$$

where $f(x)$ in the outer sum runs through all $2^{|\Lambda|} - 1$ non-empty products of polynomials $\lambda(x) \in \Lambda$. It is easily checked that the discriminant over $\mathbb{Q}$ of $\prod_{\lambda(x)\in\Lambda} \lambda(x)$ is non-zero. Hence, assuming that $p$ is sufficiently large, $f(x)$ is never a constant multiple of a square over GF$(p)$. So by Lemma 9.5.3, we have $\Delta = p + O(\sqrt{p})$, a contradiction. □

**Lemma 9.5.5** *Let $v = 3p$, $p$ prime, $p \equiv 3 \pmod 4$. Let $\Lambda$ be the set of polynomials in Lemma 9.5.4. Then there exists a polynomial $Q(x)$ such that if $\alpha \equiv 0 \pmod 3$, if $(\lambda(\alpha)/p) = 1$ for all $\lambda(x) \in \Lambda$ and if $Q(\alpha) \not\equiv 0 \pmod p$, then the STS$(v)$ of Theorem 9.5.1 generated by $\{0, 1, \alpha\}$ and $\{0, p, 2p\}$ is 6-sparse.*

**Proof.** Let $v = 3p$, $p$ prime, $p \equiv 3 \pmod 4$ and suppose $\alpha$ satisfies the conditions of the lemma with $Q(x)$ to be chosen later. Observe that $x - 1 \in \Lambda$; therefore $((\alpha - 1)/p) = 1$, as required by Theorem 9.5.1, and hence there exists a Steiner triple system $S = (V, \mathcal{B})$ generated by $\{0, 1, \alpha\}$ and $\{0, p, 2p\}$. We show that $S$ is 6-sparse.

Let $\Gamma = \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{13}$ be the set of configurations in Lemma 9.5.2. Let $\mathcal{G} \in \Gamma$ and let $\mathcal{G}$ have $n + 1$ blocks. For $d = 1, 2, \ldots, n$, let $(a_d, b_d, c_d)$ be the $d$th block of $\mathcal{G} \setminus \{\{0, 1, \alpha\}\}$ in some ordering. Then we have the following set of $3n$ linear congruences modulo $3p$ in variables $x_1, x_2, \ldots, x_n$, $m_1, m_2, \ldots, m_n$ and the variables $\omega_d$ for those $d$ where the corresponding congruences have the first alternative on the right:

$$\begin{array}{rcl}
(a_1, b_1, c_1) & \equiv & \left\{ \begin{array}{l} (m_1, m_1 + \omega_1, m_1 + \alpha\omega_1) \\ \text{or } (m_1, m_1 + p, m_1 + 2p), \end{array} \right. \\
(a_2, b_2, c_2) & \equiv & \left\{ \begin{array}{l} (m_2, m_2 + \omega_2, m_2 + \alpha\omega_2) \\ \text{or } (m_2, m_2 + p, m_2 + 2p), \end{array} \right. \\
 & \ldots, & \\
(a_n, b_n, c_n) & \equiv & \left\{ \begin{array}{l} (m_n, m_n + \omega_n, m_n + \alpha\omega_n) \\ \text{or } (m_n, m_n + p, m_n + 2p). \end{array} \right.
\end{array}$$

On eliminating the $m_d$ we have $2n$ congruences modulo $3p$:

$$\begin{array}{rcl}
a_1 - b_1 & \equiv & -\omega_1 \ \text{ or } \ -p, \\
a_1 - c_1 & \equiv & -\alpha\omega_1 \ \text{ or } \ -2p, \\
a_2 - b_2 & \equiv & -\omega_2 \ \text{ or } \ -p, \\
a_2 - c_2 & \equiv & -\alpha\omega_2 \ \text{ or } \ -2p, \\
 & \ldots, & \\
a_n - b_n & \equiv & -\omega_n \ \text{ or } \ -p, \\
a_n - c_n & \equiv & -\alpha\omega_n \ \text{ or } \ -2p.
\end{array} \tag{9.16}$$

Thus by Lemma 9.5.2, if $S$ contains a Pasch, mitre, 6-cycle or crown configuration, there exists a $\mathcal{G} \in \Gamma$ and a corresponding set of congruences (9.16) which has, for some orderings of the blocks of $\mathcal{G}$ and some choice of the alternatives on the right of (9.16), a solution modulo 3 in which all the $\omega_d$ present satisfy $\omega_d \equiv 1 \pmod 3$ and a solution modulo $p$ in which all the $\omega_d$ present satisfy $(\omega_d/p) = 1$. To show that this cannot happen, we examine each of the $12^n$ possible sets of congruences (9.16) for each configuration $\mathcal{G} \in \Gamma$. Denote this collection of congruence sets by $\Phi_0$. Thus $|\Phi_0| = 12^3 + 3 \cdot 12^4 + 9 \cdot 12^5 = 2303424$.

As an immediate first step, we eliminate from $\Phi_0$ all sets where there are two intersecting blocks in the orbit of $\{0, p, 2p\}$, for such configurations cannot occur in $S$. This leaves a collection $\Phi_1$ of 584064 congruence sets: 864 for $\mathcal{G}_1$, 7776 each for $\mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$, and 62208 each for $\mathcal{G}_5, \mathcal{G}_6, \ldots, \mathcal{G}_{13}$.

Next, we eliminate from $\Phi_1$ all cases where (9.16) has no solution modulo 3. We assume that $\alpha = 0$ and that $\omega_d = 1$ for all multipliers $\omega_d$ present. We can also assume that $p = 1$. For if a set of congruences (9.16) has a solution modulo 3 with $p = 2$ and includes the pairs $\{a_j - b_j \equiv -p,\ a_j - c_j \equiv -2p\}$ for those $j \in \{1, 2, \ldots, n\}$ where the block $\{a_j, b_j, c_j\}$ is in the orbit of $\{0, p, 2p\}$, then the set of congruences obtained by interchanging $b_j$ and $c_j$ has the same solution with $p = 1$. After performing the computations we are left with the collection $\Phi_2$ of 3320 sets, partitioned as follows: $\mathcal{G}_1$, 32; $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, 168 each; $\mathcal{G}_5, \mathcal{G}_6, \mathcal{G}_7$, 384 each; $\mathcal{G}_8, \mathcal{G}_{13}$, 344 each; $\mathcal{G}_9, \mathcal{G}_{12}$, 224 each; $\mathcal{G}_{10}, \mathcal{G}_{11}$, 248 each. In each case the solution modulo 3 is unique.

We deal with $\Phi_2$ by examining the congruence sets modulo $p$. Let $s$ be the number of blocks in the orbit of $\{0, p, 2p\}$ and note that $0 \le s \le 2$. Recall that the configuration has $n + 1$ blocks. So there are $2n$ congruences, $n$ point variables, $x_1, x_2, \ldots, x_n$, and $n - s$ multiplier variables, $\omega_d$. We select $2n - s$ congruences by excluding $s$ (possibly none) of the $2s$ congruences that involve $p$. We find that, for at least one choice of excluded congruence(s) if $s \ge 1$, the determinant of the $(2n - s) \times (2n - s)$ matrix of coefficients of the set of congruences is $d(\alpha)$ for some $d(x) \in \mathbb{Z}[x]$ which is not identically zero. Let us assume that $d(\alpha) \not\equiv 0 \pmod p$. We compute the unique solution, $\{x_1, x_2, \ldots, x_n, \ldots, \omega_d, \ldots\}$, of (9.16) over $\mathbb{Q}$, and by Lemma 9.5.1 the unique solution modulo $p$ is the same.

If $s \ge 1$ and the excluded congruences are not consistent with the other $2n - s$ congruences for all $\alpha$, then there is an additional constraint of the form $q(\alpha) \equiv 0 \pmod p$ for some $q(x) \in \mathbb{Z}[x] \setminus \{0\}$. If we assume that $q(\alpha) \not\equiv 0 \pmod p$, the configuration cannot occur in $S$.

If $s = 0$ or if $s \ge 1$ and the excluded congruences are consistent with the other $2n - s$ congruences for all $\alpha$, we attempt to compute the quadratic characters of the multipliers $\omega_j$ and the ratios $\omega_j/\omega_k$ on the assumption that $(\lambda(\alpha)/p) = 1$ for each $\lambda(x) \in \Lambda$. In all except four cases we find that at least one multiplier or ratio of multipliers is not a quadratic residue modulo $p$, and hence the corresponding configuration cannot occur in $S$. The remaining possible configurations (a Pasch and three 6-cycles) have every block congruent to $\{0, 1, \alpha\}$ modulo $p$ with $\omega_d \equiv 1 \pmod{p}$, $d = 1, 2, \ldots, n$, and it is clear that they cannot exist in $S$.

To complete the proof we set $Q(x)$ equal to the least common multiple of all the determinant polynomials $d(x)$ and constraint polynomials $q(x)$ encountered in the preceding analysis. □

**Proof of Theorem 9.5.2.** The result follows from Lemma 9.5.4 and Lemma 9.5.5. We can assume that $p$ does not divide any of the coefficients of $Q(x)$ and is sufficiently large for Lemma 9.5.4 to apply. Choosing $N$ greater than the degree of $Q(x)$, from Lemma 9.5.4 we select an $\alpha$ which is not a root of $Q(x)$ modulo $p$ and if necessary we add a multiple of $p$ to obtain a value that is congruent to 0 modulo 3. □

# Chapter 10

# Perfect and uniform Steiner triple systems

## 10.1 Introduction

For $k$ even and $k \geq 4$, a *k-cycle configuration* is a configuration of $k+2$ points consisting of the $k$ blocks

$$\{\{a,1,2\},\{b,1,3\},\{a,3,4\},\{b,2,5\},\ldots,\{a,k-1,k\},\{b,k-2,k\}\}, \tag{10.1}$$

where in this representation the points $a$ and $b$ have degree $k/2$ and all other points have degree 2. Observe that the Pasch configuration is obtained by setting $k=4$ and indeed it is often referred to as a 4-cycle.

For any two distinct points $a, b \in V$ of a Steiner triple system of order $v$, $(V, \mathcal{B})$, define the *cycle graph* $G_{a,b}$ as the graph whose point set is $V \setminus \{a, b, a*b\}$ and where $\{x, y\}$ is an edge if and only if either $\{a, x, y\} \in \mathcal{B}$ or $\{b, x, y\} \in \mathcal{B}$. It is easy to see (by drawing a diagram) that $G_{a,b}$ is a union of disjoint cycles $C_{k_1} \cup C_{k_2} \cup \cdots \cup C_{k_r}$, where $r \geq 1$, $k_i$ is even, $k_i \geq 4$, $i = 1, 2, \ldots, r$, and $k_1 + k_2 + \cdots + k_r = v - 3$. Moreover, if we take any one of the constituent cycle graphs, $C_{k_i}$, say, then the set of $k_i$ blocks

$$\{\{a,x,y\} : \{x,y\} \text{ is an edge of } C_{k_i}\} \cup \{\{b,x,y\} : \{x,y\} \text{ is an edge of } C_{k_i}\}$$

is a $k$-cycle configuration. Thus $k$-cycle configurations are unavoidable in Steiner triple systems. In fact, for each pair of points $a, b$ there will be a set of $k$-cycle configurations of the form (10.1) involving a total of $v-3$ blocks.

Two special cases are of interest. (i) A Steiner triple system where each pair of distinct points generates the same cycle graph (up to isomorphism) is called *uniform*;

and (ii) A uniform STS($v$) where each cycle graph consists of a single $(v-3)$-cycle is called *perfect*.

Doubly homogeneous Steiner triple systems are necessarily uniform. For instance, the systems of Theorem 9.2.1 with $\alpha^2 - \alpha + 1 = 0$ are doubly homogeneous and hence, as shown in [35], this construction gives immediately an infinite number of uniform systems. However there are others. Amongst the block transitive Steiner triple systems arising from Theorem 9.2.1 other than the doubly homogeneous ones, Grannell, Griggs & Murphy [35] identified several uniform systems. Those systems which are not perfect are listed in Table 10.1. We add one more, namely the system with $v = 180907$ and $\alpha = 68356$, found in October 2004. No further systems have been found for $v \leq 460000$ and $\alpha^2 - \alpha + 1 \neq 0$.

Table 10.1: Uniform block transitive Steiner triple systems

| | | | | | |
|---|---|---|---|---|---|
| $v$ | 31 | 43 | 13063 | 34303 | 180907 |
| $\alpha$ | 12 | 10 | 2174 | 5386 | 68356 |
| Cycle lengths | $\{4^7\}$ | $\{4, 36\}$ | $\{4, 13056\}$ | $\{4, 34296\}$ | $\{4, 12, 180888\}$ |
| | PG(4,2) | [35] | [35] | [35] | [25] |

Perfect systems appear to be much rarer. In contrast to uniform systems, only a finite number are known. The STS(7) and the STS(9) are perfect. There are also perfect systems for $v = 25$ and $v = 33$. The STS(25) is generated from the four starter blocks $\{(0,0),(0,1),(1,0)\}$, $\{(0,0),(0,2),(2,1)\}$, $\{0,0),(1,1),(2,3)\}$, $\{0,0),(1,3),(3,3)\}$ under the mappings $(x,y) \mapsto (x+1,y)$ and $(x,y) \mapsto (x,y+1)$ modulo 5 [69]. The STS(33) is the cyclic system with starter blocks $\{0,1,7\}$, $\{0,2,21\}$, $\{0,3,20\}$, $\{0,4,28\}$, $\{0,8,18\}$, $\{0,11,22\}$, and is also generated by the construction of Theorem 9.5.1 with $\tau = 2$ and $\alpha = 7$.

Again using the construction of Theorem 9.2.1, Grannell, Griggs & Murphy [35] identified a further nine perfect systems. We list them in Table 10.2 together with the STS(7) and one new system that we discovered, also in October 2004, namely the one with $v = 135859$ and $\alpha = 49142$. Thus there are a total of fourteen known non-trivial perfect Steiner triple systems. In the third row of the table we give the largest $k$ for which the system is $k$-sparse. Incidentally, the perfect STS(25) and the perfect STS(33) are 5-sparse but not 6-sparse.

Table 10.2: Perfect block transitive Steiner triple systems

| $v$ | 7 | 79 | 139 | 367 | 811 | 1531 | 25771 | 50923 | 61339 | 69991 | 135859 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 3 | 29 | 25 | 112 | 18 | 84 | 4525 | 12999 | 630 | 7175 | 49142 |
| $k$-sparse | | 5 | 6 | 5 | 5 | 4 | 4 | 4 | 4 | 4 | 4 |
| | | [35] | [35] | [35] | [35] | [35] | [35] | [35] | [35] | [35] | [25] |

# 10.2 12-cycles in block transitive systems

In this section we shall prove that the number of perfect Steiner triple systems that can arise from Theorem 9.2.1 is finite. We use the technique described in section 9.4 involving Weil's theorem in the form of Lemma 9.4.1 to prove that the 12-cycle configuration is unavoidable in all sufficiently large block transitive systems arising from Theorem 9.2.1 with $\alpha^2-\alpha+1 \neq 0$. The results are contained in the next two lemmas. Both lemmas involve the number $4\,290\,908\,300\,250\,625$, which we denote by $v^*$. We have tried the same approach with $k$-cycles for $k=4,6,8$ and 10 without success.

**Lemma 10.2.1** *Suppose that $S=(V,\mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. If $v>v^*$ and if all of $\alpha^2-\alpha+1$, $\alpha^2-3\alpha+1$, $1-3\alpha$, $3-\alpha$ are non-zero, then $S$ contains a 12-cycle.*

**Proof.** The blocks of the 12-cycle will be denoted by $\mathbf{b}_i$ for $i=1,2,\ldots,12$, where $\mathbf{b}_1=\{0,z_1,z_2\}$, $\mathbf{b}_2=\{a,z_2,z_3\}$, $\mathbf{b}_3=\{0,z_3,z_4\},\ldots,$ $\mathbf{b}_{12}=\{a,z_{12},z_1\}$. The point $a$ and the twelve points $z_i$ are given in terms of a parameter $x$ as follows.

$$\begin{aligned}
&a=(1-\alpha)x+\alpha,\quad z_1=1,\quad z_2=\alpha,\quad z_3=\alpha-\alpha x,\quad z_4=1-x,\\
&z_5=\frac{2(1-\alpha)x}{\alpha}+\frac{2\alpha-1}{\alpha},\quad z_6=-2x+\frac{2\alpha-1}{\alpha-1},\\
&z_7=-\frac{(\alpha+1)x}{\alpha-1}+\frac{\alpha^2}{(\alpha-1)^2},\quad z_8=-\frac{\alpha(\alpha+1)x}{\alpha-1}+\frac{\alpha^3}{(\alpha-1)^2},\\
&z_9=2\alpha x-\frac{\alpha^2}{\alpha-1},\quad z_{10}=2\alpha(1-\alpha)x+\alpha^2,\quad z_{11}=\alpha x,\quad z_{12}=x.
\end{aligned}$$

Each block $\mathbf{b}_i$ can be expressed as $\mu_i\{0,1,\alpha\}+\nu_i$ where the values of $\mu_i$ are as follows.

$$\begin{aligned}
&\mu_1=1,\quad \mu_2=x,\quad \mu_3=1-x,\quad \mu_4=\frac{(2-\alpha)x}{\alpha}+\frac{\alpha-1}{\alpha},\\
&\mu_5=\frac{2x}{\alpha}+\frac{1-2\alpha}{\alpha(\alpha-1)},\quad \mu_6=-\frac{(\alpha-3)x}{\alpha-1}+\frac{\alpha^2-3\alpha+1}{(\alpha-1)^2},\\
&\mu_7=-\frac{(\alpha+1)x}{\alpha-1}+\frac{\alpha^2}{(\alpha-1)^2},\quad \mu_8=\frac{(3\alpha-1)x}{\alpha-1}+\frac{\alpha-2\alpha^2}{(\alpha-1)^2},\\
&\mu_9=-2\alpha x+\frac{\alpha^2}{\alpha-1},\quad \mu_{10}=(1-2\alpha)x+\alpha,\quad \mu_{11}=x,\quad \mu_{12}=1-x.
\end{aligned}$$

Thus the 12 blocks will lie in the system $S$ provided that $\chi(\mu_i) = 1$ for each $i$. So take $f_i(x) = \mu_{i+1}$ for $i = 1, 2, \ldots, 9$. The reader can check that the conditions on $\alpha$ ensure that these nine functions of $x$ are polynomials of degree 1 having distinct roots $\rho_i$ with the additional property that for each $i$ there exists $j$ for which $\chi(f_j(\rho_i)) \neq 1$. Although this is lengthy and tedious, it is straightforward, and we leave the details to the reader. It may be helpful to point out that two particular conditions encountered in the checking process, namely $1 - 3\alpha + 2\alpha^2 - \alpha^3 \neq 0$ and $1 - 2\alpha + 3\alpha^2 - \alpha^3 \neq 0$, follow from the facts that neither $\chi(\alpha)$ nor $\chi(\alpha^2)$ can equal $\pm 1$, and so $\alpha \neq (\alpha - 1)^3$ and $\alpha^2 \neq (\alpha - 1)^3$. It is also necessary to verify that the 12 blocks are distinct. An effective method for doing this is to show first that $a \neq 0$. This follows from the fact that if $a = 0$ then $x = -\alpha/(1 - \alpha)$, and this leads to a contradiction between the conditions on $\alpha$ and the assumption that $\chi(x) = \chi(\mu_2) = 1$. It then follows that the six odd-numbered blocks are distinct from the six even-numbered blocks. It is also easy to show that for any $i$, $\mathbf{b}_i \neq \mathbf{b}_{i+2}$ (subscript arithmetic modulo 12), that $\mathbf{b}_i = \mathbf{b}_{i+4}$ if and only if $z_j = z_{j+4}$ for each $j$, and that $\mathbf{b}_i = \mathbf{b}_{i+6}$ if and only if $z_j = z_{j+6}$ for each $j$. Since the pair of equations $z_1 = z_5, z_2 = z_6$ lead to a contradiction as does the pair $z_1 = z_7, z_6 = z_{12}$, it then follows that the 12 blocks are distinct. Finally, applying Lemma 9.4.1, we find that a suitable $x$ may be chosen provided that $v > (6^8 \cdot 39 + 1)^2 = v^*$. □

**Lemma 10.2.2** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. If $v > v^*$ and if all of $\alpha^2 - \alpha + 1$, $\alpha^2 + \alpha - 1$, $2 - 3\alpha$, $2 + \alpha$ are non-zero, then $S$ contains a 12-cycle.*

**Proof.** The proof is similar to that of the previous lemma. We take

$$\begin{aligned}
&a = \alpha x - \frac{1}{\alpha - 1}, \quad z_1 = 1, \quad z_2 = \alpha, \quad z_3 = (\alpha - 1)x + \frac{\alpha - 1}{\alpha}, \\
&z_4 = -x - \frac{1}{\alpha}, \quad z_5 = -\frac{2\alpha x}{\alpha - 1} + \frac{2 - \alpha}{(\alpha - 1)^2}, \quad z_6 = 2\alpha x + \frac{\alpha - 2}{\alpha - 1}, \\
&z_7 = (\alpha + 1)x - \frac{1}{\alpha(\alpha - 1)}, \quad z_8 = -\frac{(\alpha + 1)x}{\alpha - 1} + \frac{1}{\alpha(\alpha - 1)^2}, \\
&z_9 = -\frac{2\alpha^2 x}{(\alpha - 1)^2} + \frac{\alpha}{(\alpha - 1)^3}, \quad z_{10} = -\frac{2\alpha x}{\alpha - 1} + \frac{1}{(\alpha - 1)^2}, \\
&z_{11} = -x, \quad z_{12} = (\alpha - 1)x.
\end{aligned}$$

These give

$$\begin{aligned}
\mu_1 &= 1, \quad \mu_2 = -x + \frac{\alpha^2-\alpha+1}{\alpha(\alpha-1)}, \quad \mu_3 = x + \frac{1}{\alpha}, \\
\mu_4 &= \frac{(\alpha+1)x}{\alpha-1} - \frac{1}{\alpha(\alpha-1)^2}, \quad \mu_5 = \frac{2\alpha x}{\alpha-1} + \frac{\alpha-2}{(\alpha-1)^2}, \\
\mu_6 &= x + \frac{1}{\alpha} = \mu_3, \quad \mu_7 = \frac{(\alpha+1)x}{\alpha-1} - \frac{1}{\alpha(\alpha-1)^2} = \mu_4, \\
\mu_8 &= \frac{(\alpha^2+1)x}{(\alpha-1)^2} - \frac{\alpha^2-\alpha+1}{\alpha(\alpha-1)^3}, \quad \mu_9 = \frac{2\alpha x}{(\alpha-1)^2} - \frac{1}{(\alpha-1)^3}, \\
\mu_{10} &= \frac{(\alpha+1)x}{\alpha-1} - \frac{1}{(\alpha-1)^2}, \quad \mu_{11} = x, \quad \mu_{12} = -x + \frac{1}{\alpha-1}.
\end{aligned}$$

Now take the nine functions $f_i(x)$ to be the expressions $\mu_j$ with $\mu_1$ and the replicated $\mu_6$ and $\mu_7$ excluded. The reader can again check that the conditions on $\alpha$ ensure that these nine functions of $x$ are polynomials of degree 1 having distinct roots $\rho_i$ with the additional property that for each $i$ there exists $j$ for which $\chi(f_j(\rho_i)) \neq 1$. To prove that the 12 blocks are distinct, we may again argue that $a \neq 0$, and to see this consider the value of $\chi(\mu_3)$ when $x = 1/\alpha(\alpha-1)$. The remainder of the argument is as in the previous lemma and, again applying Lemma 9.4.1, we find that a suitable $x$ may be chosen provided that $v > (6^8 \cdot 39 + 1)^2 = v^*$. □

**Theorem 10.2.1** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. If $v > v^*$ and if $\alpha^2 - \alpha + 1 \neq 0$, then $S$ contains a 12-cycle.*

**Proof.** This follows from Lemmas 10.2.1 and 10.2.2 once one observes that apart from $\alpha^2 - \alpha + 1 \neq 0$ the conditions on $\alpha$ in these lemmas are mutually exclusive. □

**Theorem 10.2.2** *Suppose that $S = (V, \mathcal{B})$ is a block transitive Steiner triple system of order $v$ with parameter $\alpha$. If $v > v^*$, then $S$ is not perfect.*

**Proof.** By the previous theorem, the result is true unless $\alpha^2 - \alpha + 1 = 0$. But in this exceptional case, as noted in the proof of Theorem 9.4.3, the system has either a 4-cycle or a 6-cycle. □

We have reasons for believing that the bound $v^*$ of the preceding theorem is very much too large. In our researches we have found a considerable number of 12-cycle configurations which have the same kind of unavoidance property as described by Lemmas 10.2.1 and 10.2.2, with various conditions on $\alpha$ (but always $\alpha^2 - \alpha + 1 \neq 0$). We selected the two particular 12-cycle configurations for these lemmas merely because they suited our main purpose, namely to prove Theorem 10.2.1. We have checked all systems obtained from the construction of Theorem 9.2.1 up to $v = 760\,000$ and no further perfect systems have been found. Although we are not prepared to conjecture anything, we would not be surprised if Table 10.2 is complete.

# Chapter 11

# Type B $\chi$-colourable $S(2,4,v)$ designs

## 11.1 Introduction

In this final chapter we revert to colourings, but not of Steiner triple systems. Here we consider the problem of constructing $S(2,4,v)$ Steiner systems with a special form of colouring. However, there is a connection with Steiner triple systems because, as we shall see, our constructions involve in some sense a 'stitching' together of two or more of these systems.

Recall that a *Steiner system*, $S(t,k,v)$, is a pair $(V,\mathcal{B})$ where $V$ is a set of cardinality $v$ of *elements*, or *points*, and $\mathcal{B}$ is a collection of $k$-subsets of $V$, called *blocks*, which has the property that every $t$-element subset of $V$ occurs in precisely one block. In this chapter we are concerned only with the cases $t=2$ and $k=3$ or 4. Recall that an $S(2,3,v)$ is usually called a *Steiner triple system of order* $v$, or STS($v$) for short. An STS($v$) exists if and only if $v \equiv 1$ or 3 (mod 6) [45], and an $S(2,4,v)$ exists if and only if $v \equiv 1$ or 4 (mod 12) [41]. Note that $v=1$ is admissible in both cases—the STS(1) and the $S(2,4,1)$ each consists of a single point and an empty set of blocks. A resolvable Steiner triple system is an STS($v$) whose blocks can be partitioned into $(v-1)/2$ *resolution classes* $\mathcal{B}_i$, $i=1,2,\ldots,(v-1)/2$, where $|\mathcal{B}_i| = v/3$ and $\mathcal{B}_i$ covers the entire point set. A *Kirkman triple system of order* $v$, KTS($v$), is a resolvable STS($v$) with a specified partition into resolution classes. A KTS($v$) exists if and only if $v \equiv 3$ (mod 6) [49, 59].

Let $S=(V,\mathcal{B})$ be an $S(2,4,v)$. Following Milici, Rosa and Voloshin [54], for integer $\chi \geq 2$, we define a *type B* $\chi$*-colouring* of $S$ as a surjection $\phi : V \to \Gamma$, where

$\Gamma$ is a set of cardinality $\chi$ whose elements are called *colours*, such that every block contains three points of one colour and one point of a different colour. In section 11.2 we use the colour set $\Gamma = \{\Gamma_1, \Gamma_2, \dots \Gamma_\chi\}$, and we call $\gamma_i = |\phi^{-1}(\Gamma_i)|$, $i = 1, 2, \dots, \chi$, the colour class sizes. However, in describing constructions it is more convenient to use plain letters for the elements of $\Gamma$, in which case we would, for instance, refer to the members of $\phi^{-1}(X)$ as '$X$ points'. Also it is worth mentioning that 'type B' designates one of the five distinct ways of colouring a quadruple of points, as typified by the patterns $\{A, A, A, A\}$ (type A), $\{A, A, A, B\}$ (type B), $\{A, A, B, B\}$ (type C), $\{A, A, B, C\}$ (type D) and $\{A, B, C, D\}$ (type E) [54].

## 11.2 Type B $\chi$-colourable $S(2, 4, v)$ systems

In the following lemmas we review some properties of $S(2, 4, v)$ systems and their type B colourings. The first lemma is inherent in [54].

**Lemma 11.2.1** *Suppose $S = (V, \mathcal{B})$ is an $S(2, 4, v)$ with a type B $\chi$-colouring $\phi : V \to \Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_\chi\}$. For $i = 1, 2, \dots, \chi$, let $V_i = \phi^{-1}(\Gamma_i)$; then*

$$(V_i, \{\{a, b, c\} : \{a, b, c, d\} \in \mathcal{B}, \{a, b, c\} \subseteq V_i\}) \tag{11.1}$$

*is a Steiner triple system of order $|V_i|$.*

**Proof.** Suppose $\{a, b\} \subseteq V_i$. Then $a$ and $b$ must both occur in a block of $S$ together with precisely one other point of the same colour. $\square$

**Lemma 11.2.2** *Let $S = (V, \mathcal{B})$ be an $S(2, 4, v)$ with $V = \{q_1, q_2, \dots, q_v\}$, and let $(K, \mathcal{C})$, $K \cap V = \emptyset$, be a KTS$(2v + 1)$ with resolution classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\dots$, $\mathcal{C}_v$. Let*

$$\mathcal{Q} = \bigcup_{i=1}^{v} \{\{x, y, z, q_i\} : \{x, y, z\} \in \mathcal{C}_i\}.$$

*Then $S' = (V \cup K, \mathcal{B} \cup \mathcal{Q})$ is an $S(2, 4, 3v + 1)$. Furthermore, if $S$ is type B $\chi$-colourable, then $S'$ is type B $(\chi + 1)$-colourable.*

**Proof.** This is the well-known $3v + 1$ construction. Also it is plain that assigning a $(\chi + 1)$-th colour to the points of the Kirkman triple system results in a valid type B $(\chi + 1)$-colouring of $S'$ [54]. $\square$

We will show in Theorem 11.2.1 that, by careful choice of the KTS$(2v+1)$, it is generally possible to obtain an alternative type B $(\chi+1)$-colouring pattern for the $S(2,4,3v+1)$.

**Lemma 11.2.3** *For $v=(3^\chi-1)/2$, $\chi=2,3,\dots$, there exists a type B $\chi$-colourable $S(2,4,v)$.*

**Proof.** Clearly, the $S(2,4,4)$ system has a type B 2-colouring. Apply Lemma 11.2.2 recursively to obtain systems with orders given in the following table.

| $\chi$ | 2 | 3 | 4 | 5 | 6 | 7 | ... | $\chi$ |
|---|---|---|---|---|---|---|---|---|
| $v$ | 4 | 13 | 40 | 121 | 364 | 1093 | ... | $\frac{1}{2}(3^\chi-1)$ |

For the Kirkman triple system, one can use the affine STS$(3^\chi)$ and resolution classes described in Anderson [1, pp 149–150]. □

It seems that prior to the writing of [29], Lemma 11.2.3 accounted for the only known examples of type B $\chi$-colourable $S(2,4,v)$ systems. All such systems, however, must satisfy the conditions in the next lemma.

**Lemma 11.2.4** *Let $(V,\mathcal{B})$ be a type B $\chi$-colourable $S(2,4,v)$ with colour class sizes $\gamma_1,\gamma_2,\dots,\gamma_\chi$. Then*

*(i) for $i=1,2,\dots,\chi$, $\gamma_i\equiv 1$ or $3 \pmod 6$ with exactly one $\gamma_i\equiv 1 \pmod 6$;*

*(ii) $\sum_{i=1}^{\chi}\binom{\gamma_i}{2}=\sum_{1\le i<j\le\chi}\gamma_i\gamma_j=\frac{1}{4}v(v-1)$;*

*(iii) for $i=1,2,\dots,\chi$, $\gamma_i\le\frac{1}{3}(2v+1)$;*

*(iv) $\gamma_i=\frac{1}{3}(2v+1)$ for some $i$ if and only if the $S(2,4,v)$ can be obtained from an $S(2,4,v-\gamma_i)$ via Lemma 11.2.2;*

*(v) for $0\le i<j\le k$, $(\gamma_i-\gamma_j)^2\ge\gamma_i+\gamma_j$.*

**Proof.** For items (i)–(iii), see Lemma 3.7 of [54], and (iv) follows easily from the proof of Lemma 11.2.2 (above). For (v), denote the $i$-th colour class by $\Gamma_i$ and observe that the $\gamma_i\gamma_j$ $\{\Gamma_i,\Gamma_j\}$ pairs, $i\ne j$, must come from blocks of the form $\{\Gamma_i,\Gamma_i,\Gamma_i,\Gamma_j\}$ or $\{\Gamma_j,\Gamma_j,\Gamma_j,\Gamma_i\}$. Hence

$$\frac{1}{2}\gamma_i(\gamma_i-1)+\frac{1}{2}\gamma_j(\gamma_j-1)\ge\gamma_i\gamma_j.$$

Observe in particular that one cannot have two equal colour class sizes. □

For $v \leq 505$, the only possible parameter sets satisfying Lemma 11.2.4, other than those arising from Lemma 11.2.3, are given in Table 11.1. When $\chi = 2$ we can characterize the possible values of $v$ precisely. For inequality (v) in Lemma 11.2.4 becomes an equality, $(\gamma_1 - \gamma_2)^2 = \gamma_1 + \gamma_2$, which, together with $v = \gamma_1 + \gamma_2$, yields

$$\gamma_1 = \frac{v \pm \sqrt{v}}{2} \equiv 1, 3 \pmod 6;$$

hence $v = (12s+2)^2$ or $(12s+10)^2$, $s = 0, 1, 2, \ldots$.

Table 11.1: Parameters of possible type B $\chi$-colourable $S(2,4,v)$s

| $v$ | $\chi$ | $\gamma_1, \gamma_2, \ldots, \gamma_\chi$ | $v$ | $\chi$ | $\gamma_1, \gamma_2, \ldots, \gamma_\chi$ |
|---|---|---|---|---|---|
| 61 | 3 | 3, 19, 39 | 313 | 5 | 1, 3, 9, 105, 195 |
| 100 | 2 | 45, 55 | 328 | 4 | 1, 3, 135, 189 |
| 109 | 3 | 1, 45, 63 | 328 | 4 | 1, 45, 63, 219 |
| 184 | 4 | 1, 9, 57, 117 | 328 | 4 | 9, 15, 91, 213 |
| 184 | 4 | 3, 19, 39, 123 | 361 | 5 | 3, 9, 15, 99, 235 |
| 196 | 2 | 91, 105 | 361 | 5 | 1, 9, 21, 93, 237 |
| 232 | 4 | 3, 9, 73, 147 | 397 | 3 | 19, 129, 249 |
| 232 | 4 | 3, 19, 57, 153 | 424 | 4 | 9, 19, 123, 273 |
| 301 | 3 | 1, 135, 165 | 457 | 5 | 3, 15, 27, 109, 303 |
| 301 | 3 | 9, 109, 183 | 484 | 2 | 231, 253 |
| 301 | 3 | 33, 69, 199 | 505 | 5 | 3, 9, 21, 147, 325 |
| 301 | 3 | 45, 55, 201 | 505 | 5 | 9, 15, 21, 127, 333 |

In the following sections we give constructions for the first four systems listed in Table 11.1, $v = 61$, 100, 109 and 184. We do not claim that any of the systems are unique up to isomorphism. Indeed, for $v = 109$, which we found to be the easiest to construct, we have several pairwise non-isomorphic examples. Further systems arise from repeated use of Theorem 11.2.1, below.

Theorem 11.2.1 may be regarded as an extension of Lemma 11.2.2, where the KTS$(2v+1)$ is obtained from the $S(2,4,v)$ and the resulting $S(2,4,3v+1)$ then generally has two alternative type B $(\chi+1)$-colouring patterns.

**Theorem 11.2.1** *Let $S$ be a type B $\chi$-colourable $S(2,4,v)$ system with colour class sizes $\{\gamma_1, \gamma_2, \ldots, \gamma_\chi\}$. Then there exists a type B $(\chi+1)$-colourable $S(2,4,3v+1)$*

*which may be coloured either with colour class sizes* $\{\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_\chi,\ 2v+1\}$ *or with colour class sizes* $\{3\gamma_1,\ 3\gamma_2,\ \ldots,\ 3\gamma_\chi,\ 1\}$.

**Proof.** The former colouring pattern is generated by Lemma 11.2.2 using any KTS$(2v+1)$ in that construction.

To deal with the latter colouring pattern, let $S = (V, \mathcal{B})$, where $V = \{i_0 : i = 0, 1, \ldots, v-1\}$. For each block $\{x_0, y_0, z_0, w_0\} \in \mathcal{B}$, take a fixed ordering of the block, $(x_0, y_0, z_0, w_0)$. From these ordered blocks we create a KTS$(2v+1)$ on the point set $V' = \{i_1, i_2 : i = 0, 1, \ldots, v-1\} \cup \{\infty\}$. We list the blocks of this design in $v$ parallel classes each of which is associated with a single point of $V$. The ordered block $(x_0, y_0, z_0, w_0)$ obtained from $\mathcal{B}$ contributes the following triples to these classes.

(i) $\{y_1, z_2, w_1\}$ and $\{y_2, z_1, w_2\}$ associated with $x_0$,

(ii) $\{x_2, z_1, w_1\}$ and $\{x_1, z_2, w_2\}$ associated with $y_0$,

(iii) $\{x_1, y_2, w_1\}$ and $\{x_2, y_1, w_2\}$ associated with $z_0$,

(iv) $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ associated with $w_0$.

In addition, the class associated with $i_0$ contains the triple $\{\infty, i_1, i_2\}$. Thus each class contains $2(v-1)/3 + 1 = (2v+1)/3$ disjoint blocks and so forms a parallel class of triples on $V'$. It is also easy to see that the complete set of triples forms an STS$(2v+1)$ and hence, with the specified resolution, a KTS$(2v+1)$. From the KTS$(2v+1)$ and the original $S(2,4,v)$ we form an $S(2,4,3v+1)$ using the method of Lemma 11.2.2 and taking care to adjoin to each parallel class the point of $V$ with which it is associated. We now colour the points $i_1, i_2$ with the same colour as $i_0$ for each $i = 0, 1, \ldots, v-1$ and we assign a new colour to the point $\infty$.

We note that the two colour patterns presented are identical if and only if $\{\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_\chi,\ 2v+1\} = \{1,\ 3\gamma_1,\ 3\gamma_2,\ \ldots,\ 3\gamma_\chi\} = \{1,\ 3,\ 3^2,\ \ldots,\ 3^\chi\}$, which is the case covered by Lemma 11.2.3. □

## 11.3 A type B 3-colourable $S(2,4,61)$

Here we consider the first entry in Table 11.1 by constructing an $S(2,4,61)$ together with a type B 3-colouring having colour class sizes 39, 19 and 3. Denote the corresponding colours by $A$, $B$ and $C$, respectively. For each of the three STS($v$)s identified by Lemma 11.2.1, let the points be the integers $0, 1, \ldots, v-1$ indexed by the system's colour. We refer to a point by any of the descriptions $X_n$, $Xn$ and $n$, where $X$ is the colour and $n$ is the integer; the second option appears in tables and the third option is used only if the colour is clear from the context. Arithmetic on points is performed on the integer parts in an appropriate ring.

Let the STS(39) have the automorphism $\alpha$ defined by

$$\alpha : A_i \mapsto A_{i+13 \pmod{39}}.$$

For the $B$ system, we choose the cyclic STS(19) with starter blocks $\{0,1,4\}$, $\{0,7,9\}$ and $\{0,11,6\}$ and automorphism $\beta$ defined by

$$\beta : B_j \mapsto B_{7j \pmod{19}}.$$

Note that $\beta$ leaves $B_0$ fixed and partitions the other $B$ points into six orbits of size 3.

We begin with the block

$$\{C_0, C_1, C_2, B_0\}$$

and we assign blocks of the $B$ system to $A$ and $C$ points as in Table 11.2. Observe that if $\{B_x, B_y, B_z\}$ is assigned to point $A_i$, then $\beta(\{B_x, B_y, B_z\})$ is assigned to point $\alpha(A_i)$, while if $\{B_x, B_y, B_z\}$ is assigned to point $C_i$ then $\beta(\{B_x, B_y, B_z\})$ is also assigned to the point $C_i$. This latter assignment is also done in such a way that each $C_i$ is paired with each $B_j, j \neq 0$. Thus we have dealt with the 171 $BB$ pairs, the 57 $BC$ pairs and the three $CC$ pairs.

For the STS(39), we first create the set of ten $A$ blocks

$$\mathcal{U}_0 = \{\{A_i, A_{13+i}, A_{26+i}\} : i = 1,2,3,4,5,6,7,8,9,11\}.$$

and assign them to $B_0$. These blocks are fixed under the action of $\alpha$ and they account for the remaining $AB_0$ pairs. For $j = 1, 2, \ldots 18$, let $\Omega_j$ denote the set of $A$

Table 11.2: STS(19) – the $B$ system for the $S(2,4,61)$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | $A0$ | 0 | 7 | 9 | $A13$ | 0 | 11 | 6 | $A26$ |
| 1 | 2 | 5 | $A1$ | 1 | 8 | 10 | $A20$ | 1 | 12 | 7 | $A30$ |
| 2 | 3 | 6 | $A2$ | 2 | 9 | 11 | $C0$ | 2 | 13 | 8 | $A35$ |
| 3 | 4 | 7 | $C0$ | 3 | 10 | 12 | $A22$ | 3 | 14 | 9 | $A28$ |
| 4 | 5 | 8 | $C1$ | 4 | 11 | 13 | $C2$ | 4 | 15 | 10 | $A32$ |
| 5 | 6 | 9 | $A3$ | 5 | 12 | 14 | $C2$ | 5 | 16 | 11 | $A37$ |
| 6 | 7 | 10 | $C2$ | 6 | 13 | 15 | $A19$ | 6 | 17 | 12 | $C1$ |
| 7 | 8 | 11 | $A4$ | 7 | 14 | 16 | $A14$ | 7 | 18 | 13 | $A33$ |
| 8 | 9 | 12 | $A5$ | 8 | 15 | 17 | $C0$ | 8 | 0 | 14 | $A38$ |
| 9 | 10 | 13 | $A6$ | 9 | 16 | 18 | $C1$ | 9 | 1 | 15 | $C2$ |
| 10 | 11 | 14 | $C1$ | 10 | 17 | 0 | $A23$ | 10 | 2 | 16 | $A34$ |
| 11 | 12 | 15 | $A7$ | 11 | 18 | 1 | $A17$ | 11 | 3 | 17 | $A27$ |
| 12 | 13 | 16 | $C0$ | 12 | 0 | 2 | $A25$ | 12 | 4 | 18 | $A31$ |
| 13 | 14 | 17 | $A8$ | 13 | 1 | 3 | $C1$ | 13 | 5 | 0 | $A36$ |
| 14 | 15 | 18 | $A9$ | 14 | 2 | 4 | $A15$ | 14 | 6 | 1 | $C0$ |
| 15 | 16 | 0 | $A10$ | 15 | 3 | 5 | $A21$ | 15 | 7 | 2 | $C1$ |
| 16 | 17 | 1 | $A11$ | 16 | 4 | 6 | $A16$ | 16 | 8 | 3 | $C2$ |
| 17 | 18 | 2 | $C2$ | 17 | 5 | 7 | $A24$ | 17 | 9 | 4 | $A29$ |
| 18 | 0 | 3 | $A12$ | 18 | 6 | 8 | $A18$ | 18 | 10 | 5 | $C0$ |

points that have so far been paired with $B_j$. Then from Table 11.2 we have

$$\begin{aligned}
\Omega_1 &= \{A_0, A_1, A_{11}, A_{17}, A_{20}, A_{30}\}, \\
\Omega_2 &= \{A_1, A_2, A_{15}, A_{25}, A_{34}, A_{35}\}, \\
\Omega_4 &= \{A_0, A_{15}, A_{16}, A_{29}, A_{31}, A_{32}\}, \\
\Omega_8 &= \{A_4, A_5, A_{18}, A_{20}, A_{35}, A_{38}\}, \\
\Omega_{16} &= \{A_{10}, A_{11}, A_{14}, A_{16}, A_{34}, A_{37}\}, \\
\Omega_{13} &= \{A_6, A_8, A_{19}, A_{33}, A_{35}, A_{36}\},
\end{aligned}$$

and $\Omega_{\beta(j)} = \alpha(\Omega_j)$.

Next we search for a set $\mathcal{T}$ of $A$ blocks such that (i) the blocks of $\mathcal{T}$ form a parallel class, (ii) no two blocks of $\mathcal{T}$ lie in the same orbit under the action of $\alpha$, and (iii) blocks which are fixed by $\alpha$ are not used in $\mathcal{T}$. A computer search produces

1038 sets, of which we select one,

$$\begin{aligned}\mathcal{T}_0 &= \{\{26,27,4\},\{11,20,30\},\{1,2,25\},\{15,34,35\},\{0,16,32\},\\ &\quad \{28,3,5\},\{17,31,33\},\{18,9,12\},\{10,14,37\},\\ &\quad \{24,29,8\},\{19,21,7\},\{6,22,23\},\{13,36,38\}\},\end{aligned}$$

and we assign the blocks of $\mathcal{T}_0$ to $C_0$. Together with $\mathcal{T}_1 = \alpha(\mathcal{T}_0)$ (assigned to $C_1$), $\mathcal{T}_2 = \alpha^2(\mathcal{T}_0)$ (assigned to $C_2$) and $\mathcal{U}_0$, we now have 49 $A$ blocks and have dealt with the 117 $AC$ pairs.

The $A$ system is extended to an STS(39) by hill climbing [68] and $A$ blocks are assigned to points $B_j$, $j = 1, 2, 4, 8, 16, 13$, under the following conditions.

(i) During the hill-climbing process, blocks are added to the system or removed from the system in triples: $X$, $\alpha(X)$ and $\alpha^2(X)$. Blocks in $\mathcal{U}_0 \cup \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ are never removed. Blocks fixed by $\alpha$ are never added.

(ii) The blocks assigned to $B_j$ form a partial parallel class $\mathcal{U}_j$ of size 11 that does not contain any of the points in $\Omega_j$.

(iii) At most one block in each orbit under the action of $\alpha$ is assigned to $B_j$.

A solution is given by assigning the blocks of $\mathcal{U}_j$ to $B_j$, $j = 1$, 2, 4, 8, 16, 13, where the $\mathcal{U}_j$ are defined by

$$\begin{aligned}\mathcal{U}_1 &= \{\{2,12,36\},\{3,22,27\},\{4,19,25\},\{5,9,13\},\\ &\quad \{6,15,37\},\{7,16,38\},\{8,10,18\},\{14,21,33\},\\ &\quad \{23,29,34\},\{24,28,31\},\{26,32,35\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_2 &= \{\{0,13,37\},\{3,9,24\},\{4,16,36\},\{5,19,33\},\\ &\quad \{6,14,18\},\{7,28,29\},\{8,31,32\},\{10,22,30\},\\ &\quad \{11,12,17\},\{20,23,26\},\{21,27,38\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_4 &= \{\{1,19,37\},\{2,33,34\},\{3,6,35\},\{4,12,14\},\\ &\quad \{5,11,22\},\{7,23,24\},\{8,27,30\},\{9,21,26\},\\ &\quad \{10,28,36\},\{13,17,20\},\{18,25,38\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_8 &= \{\{0,3,10\},\{1,6,11\},\{2,17,26\},\{7,31,36\},\\ &\quad \{8,23,37\},\{9,27,33\},\{12,22,24\},\{13,25,34\},\\ &\quad \{14,16,28\},\{15,21,32\},\{19,29,30\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_{16} &= \{\{0,5,27\},\{1,9,23\},\{2,13,31\},\{3,17,18\},\\ &\quad \{4,8,22\},\{6,7,26\},\{12,15,19\},\{20,24,38\},\\ &\quad \{21,25,29\},\{28,33,35\},\{30,32,36\}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}_{13} &= \{\{0,29,38\},\{1,10,31\},\{2,4,11\},\{3,7,14\},\\ &\quad \{5,16,24\},\{9,20,25\},\{12,23,32\},\{13,15,27\},\\ &\quad \{17,22,28\},\{18,21,30\},\{26,34,37\}\}.\end{aligned}$$

To complete the construction we assign the blocks of $\alpha(\mathcal{U}_j)$ to $\beta(B_j)$ and the blocks of $\alpha^2(\mathcal{U}_j)$ to $\alpha^2(B_j)$, $j = 1,2,4,8,16,13$. The entire $S(2,4,61)$ is listed in Table 11.3.

The $\mathcal{U}_j$ listed above were discovered by the author of this thesis on 25 March 2005, the 21st birthday of his youngest daughter. Thereafter he has always referred to the resulting type-B 3-colourable $S(2,4,61)$ as 'Zoe's design'. It first appeared on the front cover of [51].

## 11.4 A type B 2-colourable $S(2,4,100)$

Viewed from a slightly different perspective, the system described in this section, 'The Design of the Century', is the same as the $S(2,4,100)$ which is the subject of [29]. Using the same conventions as in section 11.3, let the colours corresponding to colour class sizes $(55,45)$ be $(A,B)$. Let the $S(2,4,100)$ have the automorphism $\sigma$ defined by

$$\sigma : A_i \mapsto A_{i+5 \pmod{55}}, B_j \mapsto B_{j+4 \pmod{44}}, j = 0,1,\ldots,43, B_{44} \mapsto B_{44}.$$

The $A$ system is an STS(55) and the $B$ system is an STS(45). Both systems have automorphism $\sigma$. As an aside, we remark that the $B$ system is an example of a *4-rotational* STS($v$) and that such systems exist for all $v \equiv 1,9,13$ or $21 \pmod{24}$ [9].

Blocks in the $B$ system are assigned to points $A_i$, $i = 0, 1, \ldots, 4$, subject to the conditions: (i) the $B$ blocks assigned to $A_i$ form a partial parallel class of size 6, and (ii) no more than one block in an orbit of $\sigma$ is assigned to $A_i$. Then we apply $\sigma$ to assign the rest of the blocks: if $\{B_x, B_y, B_z\}$ is assigned to $A_k$, then $\sigma(\{B_x, B_y, B_z\})$ is assigned to $\sigma(A_k)$.

Eleven $A$ blocks in a single orbit under the action of $\sigma$ are assigned to $B_{44}$ such that the points in these $A$ blocks together with the $A$ points to which the $B$ blocks containing $B_{44}$ are already assigned cover a complete set of residues modulo 5. Thus all $AB_{44}$ pairs are accounted for.

Then blocks in the $A$ system are assigned to $B_x$, $x = 0, 1, 2, 3$, subject to the conditions: (i) the $A$ blocks assigned to $B_x$ form a partial parallel class of size 11 which contains none of the 22 $A$ points to which the $B$ blocks containing $B_x$ have already been assigned, and (ii) no more than one block in an orbit of $\sigma$ is assigned to $B_x$. We complete the assignment by applying $\sigma$: if $\{A_i, A_j, A_k\}$ is assigned to $B_z$, then $\sigma(\{A_i, A_j, A_k\})$ is assigned to $\sigma(B_z)$.

Orbit representatives under $\sigma$ of the blocks of the $S(2,4,100)$ are listed in Table 11.4. A similar array appeared on the front cover of [50].

## 11.5 A type B 3-colourable $S(2,4,109)$

Let the colours corresponding to colour class sizes $(63, 45, 1)$ be $(A, B, C)$ and let the S(2,4,109) have the automorphism $\tau$ defined by

$$\tau : \begin{cases} A_i \mapsto A_{i+3 \pmod{63}}, & i = 0, 1, \ldots, 62, \\ B_j \mapsto B_{j+2 \pmod{42}}, & j = 0, 1, \ldots, 41, \\ B_{42} \mapsto B_{43} \mapsto B_{44} \mapsto B_{42}, \\ C_0 \mapsto C_0. \end{cases}$$

The $B$ system is an STS(45) containing the blocks

$$\{\{B_n, B_{n+14}, B_{n+28}\} : n = 0, 1, \ldots, 13\} \text{ and } \{B_{42}, B_{43}, B_{44}\},$$

which we assign to the point $C_0$. The remaining blocks are partitioned into 15 21-block orbits by $\tau$. We deal with these 315 blocks by assigning $B$ blocks to $A_i$, $i = 0, 1, 2$, such that (i) the $B$ blocks assigned to $A_i$ form a partial parallel class of size 5, and (ii) no more than one block in an orbit of $\tau$ is assigned to $A_i$. The assignment is then completed by applying the automorphism $\tau$.

The $A$ system is an STS(63) whose blocks are partitioned by $\tau$ into 31 orbits of size 21, at least one of which is a parallel class. We assign one of these parallel classes to $C_0$.

In the STS(45), the fifteen 21-block orbits under $\tau$ collectively contain each $B$ point 21 times. The points $B_{42}, B_{43}, B_{44}$ must each occur precisely seven times in three of these orbits. Hence there are precisely three distinct $A$ points, $x_1, x_2, x_3$, paired with $B_{42}$ such that $0 \leq x_1, x_2, x_3 < 9$. In the $A$ system we choose two blocks from different orbits, $\{x_4, x_5, x_6\}$ and $\{x_7, x_8, x_9\}$, such that $\{x_1, x_2, \ldots, x_9\}$ covers a complete set of residues modulo 9. We assign these blocks to $B_{42}$ and extend the assignment in the usual manner by applying $\tau$.

For the remaining 588 blocks in the STS(63), we assign $A$ blocks to $B_x$, $x = 0, 1$, such that (i) the $A$ blocks assigned to $B_x$ form a partial parallel class of size 14 which contains none of the 21 $A$ points to which the $B$ blocks containing $B_x$ have already been assigned, and (ii) no more than one block in an orbit of $\tau$ is assigned to $B_x$. Finally, the assignment is completed by applying $\tau$. An $S(2,4,109)$ is presented as Table 11.5.

## 11.6 A type B 4-colourable $S(2,4,184)$

Type B 4-colourable $S(2,4,184)$s with both permissible colour patterns can be constructed from the $S(2,4,61)$ given in section 11.3 using Theorem 11.2.1. Here we describe a direct construction for one of these.

The colour class sizes are $(117, 57, 9, 1)$ and we denote the corresponding colours by $(A, B, C, D)$. Let the system have the automorphism $\phi$ of order 57 defined by

$$\phi : \begin{cases} A_i \mapsto A_{i+2 \pmod{114}},\ i = 0, 1, \ldots, 113, \\ A_{114} \mapsto A_{114}, A_{115} \mapsto A_{115}, A_{116} \mapsto A_{116}, \\ B_j \mapsto B_{j+1 \pmod{57}},\ j = 0, 1, \ldots, 56, \\ C_0 \mapsto C_1 \mapsto C_2 \mapsto C_0, C_3 \mapsto C_4 \mapsto C_5 \mapsto C_3, C_6 \mapsto C_7 \mapsto C_8 \mapsto C_6, \\ D_0 \mapsto D_0. \end{cases}$$

We refer to a point $A_i$ as an $A^-$ point if $i \leq 113$ and an $A^+$ point if $i \geq 114$.

The $B$ system is a cyclic STS(57) whose blocks are partitioned into the following: (i) a short orbit with starter block $\{B_0, B_{19}, B_{38}\}$, which we assign to $D_0$; (ii) three 57-block orbits with starter blocks that each cover all three residues modulo 3; and (iii) another six 57-block orbits. The starter blocks described in (ii) are assigned to points $C_0$, $C_3$ and $C_6$, one starter block to each point. We select a representative block from each of the six orbits described in (iii) and partition them into two 3-block partial parallel classes. The blocks of one partial parallel class are assigned to $A_0$ and the blocks of the other are assigned to $A_1$.

The $A$ system is an STS(117) whose blocks are partitioned into the following: (i) a single block, $\{A_{114}, A_{115}, A_{116}\}$, which we assign to $D_0$; (ii) two 19-block orbits with starter blocks $\{A_0, A_{38}, A_{76}\}$ and $\{A_1, A_{39}, A_{77}\}$, which we also assign to $D_0$; (iii) three pairs of 57-block orbits of blocks containing only $A^-$ points where each pair of starter blocks covers all six residues modulo 6; and (iv) another thirty-three 57-block orbits. The pairs of starter blocks described in (iii) are assigned to points $C_0$, $C_3$ and $C_6$, one pair to each point. Using one block from each of the 33 orbits described in (iv), we attempt to construct a 33-block partial parallel class. If successful, we assign blocks of the partial parallel class to $B_0$.

The $C$ system is an STS(9) whose blocks are assigned thus: $\{C_0, C_1, C_2\}$, $\{C_3, C_4, C_5\}$ and $\{C_3, C_4, C_5\}$ to $D_0$, $\{C_0, C_3, C_6\}$ to $A_{114}$, $\{C_0, C_4, C_8\}$ to $A_{115}$ and $\{C_0, C_5, C_7\}$ to $A_{116}$.

Finally, the assignment of blocks to points is completed by the application of $\phi$. Observe that the 13 blocks consisting of three $C$ points or three $A^+$ points together with the points to which they are assigned form an $S(2,4,13)$ system.

An $S(2,4,184)$ is presented as Table 11.6.

Table 11.3: $S(2,4,61)$ – *Zoe's design*

| | | | | | |
|---|---|---|---|---|---|
| A1 A14 A27 B0 | A2 A15 A28 B0 | A3 A16 A29 B0 | A4 A17 A30 B0 | A5 A18 A31 B0 | A6 A19 A32 B0 |
| A7 A20 A33 B0 | A8 A21 A34 B0 | A9 A22 A35 B0 | A11 A24 A37 B0 | | |
| A0 A2 A14 B10 | A13 A15 A27 B13 | A26 A28 A1 B15 | A0 A3 A10 B8 | A13 A16 A23 B18 | A26 A29 A36 B12 |
| A0 A4 A7 B6 | A13 A17 A20 B4 | A26 A30 A33 B9 | A0 A5 A27 B16 | A13 A18 A1 B17 | A26 A31 A14 B5 |
| A0 A6 A9 B7 | A13 A19 A22 B11 | A26 A32 A35 B1 | A0 A8 A11 B15 | A13 A21 A24 B10 | A26 A34 A37 B13 |
| A0 A12 A21 B12 | A13 A25 A34 B8 | A26 A38 A8 B18 | A0 A13 A37 B2 | A13 A26 A11 B14 | A26 A0 A24 B3 |
| A0 A15 A30 B18 | A13 A28 A4 B12 | A26 A2 A17 B8 | A0 A18 A28 B5 | A13 A31 A2 B16 | A26 A5 A15 B17 |
| A0 A19 A20 B17 | A13 A32 A33 B5 | A26 A6 A7 B16 | A0 A22 A34 B9 | A13 A35 A8 B6 | A26 A9 A21 B4 |
| A0 A29 A38 B13 | A13 A3 A12 B15 | A26 A16 A25 B10 | A0 A31 A35 B11 | A13 A5 A9 B1 | A26 A18 A22 B7 |
| A0 A33 A36 B14 | A13 A7 A10 B3 | A26 A20 A23 B2 | A1 A3 A15 B12 | A14 A16 A28 B8 | A27 A29 A2 B18 |
| A1 A4 A21 B9 | A14 A17 A34 B6 | A27 A30 A8 B4 | A1 A5 A32 B3 | A14 A18 A6 B2 | A27 A31 A19 B14 |
| A1 A6 A11 B8 | A14 A19 A24 B18 | A27 A32 A37 B12 | A1 A7 A22 B18 | A14 A20 A35 B12 | A27 A33 A9 B8 |
| A1 A8 A20 B11 | A14 A21 A33 B1 | A27 A34 A7 B7 | A1 A9 A23 B16 | A14 A22 A36 B17 | A27 A35 A10 B5 |
| A1 A10 A31 B13 | A14 A23 A5 B15 | A27 A36 A18 B10 | A1 A12 A34 B14 | A14 A25 A8 B3 | A27 A38 A21 B2 |
| A1 A16 A35 B7 | A14 A29 A9 B11 | A27 A3 A22 B1 | A1 A19 A37 B4 | A14 A32 A11 B9 | A27 A6 A24 B6 |
| A1 A29 A33 B10 | A14 A3 A7 B13 | A27 A16 A20 B15 | A1 A30 A38 B6 | A14 A4 A12 B4 | A27 A17 A25 B9 |
| A2 A3 A20 B14 | A15 A16 A33 B3 | A28 A29 A7 B2 | A2 A4 A11 B13 | A15 A17 A24 B15 | A28 A30 A37 B10 |
| A2 A5 A37 B7 | A15 A18 A11 B11 | A28 A31 A24 B1 | A2 A6 A38 B5 | A15 A19 A12 B16 | A28 A32 A25 B17 |
| A2 A7 A9 B17 | A15 A20 A22 B5 | A28 A33 A35 B16 | A2 A8 A19 B12 | A15 A21 A32 B8 | A28 A34 A6 B18 |
| A2 A10 A23 B9 | A15 A23 A36 B6 | A28 A36 A10 B4 | A2 A12 A36 B1 | A15 A25 A10 B7 | A28 A38 A23 B11 |
| A2 A24 A32 B11 | A15 A37 A6 B1 | A28 A11 A19 B7 | A2 A30 A35 B15 | A15 A4 A9 B10 | A28 A17 A22 B13 |
| A2 A33 A34 B4 | A15 A7 A8 B9 | A28 A20 A21 B6 | A3 A4 A32 B18 | A16 A17 A6 B12 | A29 A30 A19 B8 |
| A3 A6 A35 B4 | A16 A19 A9 B9 | A29 A32 A22 B6 | A3 A8 A36 B7 | A16 A21 A10 B11 | A29 A34 A23 B1 |
| A3 A9 A24 B2 | A16 A22 A37 B14 | A29 A35 A11 B3 | A3 A11 A31 B10 | A16 A24 A5 B13 | A29 A37 A18 B15 |
| A3 A17 A18 B16 | A16 A30 A31 B17 | A29 A4 A5 B5 | A3 A23 A30 B3 | A16 A36 A4 B2 | A29 A10 A17 B14 |
| A3 A25 A33 B11 | A16 A38 A7 B1 | A29 A12 A20 B7 | A3 A34 A38 B17 | A16 A8 A12 B5 | A29 A21 A25 B16 |
| A4 A6 A10 B17 | A17 A19 A23 B5 | A30 A32 A36 B16 | A4 A8 A22 B16 | A17 A21 A35 B17 | A30 A34 A9 B5 |
| A4 A19 A25 B1 | A17 A32 A38 B7 | A30 A6 A12 B11 | A4 A23 A35 B14 | A17 A36 A9 B3 | A30 A10 A22 B2 |
| A4 A31 A34 B15 | A17 A5 A8 B10 | A30 A18 A21 B13 | A4 A37 A38 B3 | A17 A11 A12 B2 | A30 A24 A25 B14 |
| A5 A6 A21 B14 | A18 A19 A34 B3 | A31 A32 A8 B2 | A5 A10 A20 B18 | A18 A23 A33 B12 | A31 A36 A7 B8 |
| A5 A11 A22 B4 | A18 A24 A35 B9 | A31 A37 A9 B6 | A5 A12 A25 B6 | A18 A25 A38 B4 | A31 A38 A12 B9 |
| A5 A19 A33 B2 | A18 A32 A7 B14 | A31 A6 A20 B3 | A5 A34 A36 B11 | A18 A8 A10 B1 | A31 A21 A23 B7 |
| A6 A25 A36 B15 | A19 A38 A10 B10 | A32 A12 A23 B13 | A7 A11 A25 B5 | A20 A24 A38 B16 | A33 A37 A12 B17 |
| A7 A12 A35 B10 | A20 A25 A9 B13 | A33 A38 A22 B15 | A7 A23 A24 B4 | A20 A36 A37 B9 | A33 A10 A11 B6 |
| A8 A23 A37 B8 | A21 A36 A11 B18 | A34 A10 A24 B12 | A9 A11 A38 B12 | A22 A24 A12 B8 | A35 A37 A25 B18 |
| A26 A27 A4 C0 | A11 A20 A30 C0 | A1 A2 A25 C0 | A15 A34 A35 C0 | A0 A16 A32 C0 | A28 A3 A5 C0 |
| A17 A31 A33 C0 | A18 A9 A12 C0 | A10 A14 A37 C0 | A24 A29 A8 C0 | A19 A21 A7 C0 | A6 A22 A23 C0 |
| A13 A36 A38 C0 | | | | | |
| A0 A1 A17 C1 | A24 A33 A4 C1 | A14 A15 A38 C1 | A28 A8 A9 C1 | A13 A29 A6 C1 | A2 A16 A18 C1 |
| A30 A5 A7 C1 | A31 A22 A25 C1 | A23 A27 A11 C1 | A37 A3 A21 C1 | A32 A34 A20 C1 | A19 A35 A36 C1 |
| A10 A12 A26 C1 | | | | | |
| A13 A14 A30 C2 | A37 A7 A17 C2 | A27 A28 A12 C2 | A2 A21 A22 C2 | A26 A3 A19 C2 | A15 A29 A31 C2 |
| A4 A18 A20 C2 | A5 A35 A38 C2 | A36 A1 A24 C2 | A11 A16 A34 C2 | A6 A8 A33 C2 | A32 A9 A10 C2 |
| A0 A23 A25 C2 | | | | | C0 C1 C2 B0 |
| B0 B1 B4 A0 | B0 B7 B9 A13 | B0 B11 B6 A26 | B1 B2 B5 A1 | B1 B8 B10 A20 | B1 B12 B7 A30 |
| B2 B3 B6 A2 | B2 B13 B8 A35 | B3 B10 B12 A22 | B3 B14 B9 A28 | B4 B15 B10 A32 | B5 B6 B9 A3 |
| B5 B16 B11 A37 | B6 B13 B15 A19 | B7 B8 B11 A4 | B7 B14 B16 A14 | B7 B18 B13 A33 | B8 B9 B12 A5 |
| B8 B0 B14 A38 | B9 B10 B13 A6 | B10 B17 B0 A23 | B10 B2 B16 A34 | B11 B12 B15 A7 | B11 B18 B1 A17 |
| B11 B3 B17 A27 | B12 B0 B2 A25 | B12 B4 B18 A31 | B13 B14 B17 A8 | B13 B5 B0 A36 | B14 B15 B18 A9 |
| B14 B2 B4 A15 | B15 B16 B0 A10 | B15 B3 B5 A21 | B16 B17 B1 A11 | B16 B4 B6 A16 | B17 B5 B7 A24 |
| B17 B9 B4 A29 | B18 B0 B3 A12 | B18 B6 B8 A18 | | | |
| B2 B9 B11 C0 | B3 B4 B7 C0 | B8 B15 B17 C0 | B12 B13 B16 C0 | B14 B6 B1 C0 | B18 B10 B5 C0 |
| B4 B5 B8 C1 | B6 B17 B12 C1 | B9 B16 B18 C1 | B10 B11 B14 C1 | B13 B1 B3 C1 | B15 B7 B2 C1 |
| B4 B11 B13 C2 | B5 B12 B14 C2 | B6 B7 B10 C2 | B9 B1 B15 C2 | B16 B8 B3 C2 | B17 B18 B2 C2 |

Table 11.4: $S(2,4,100)$ – *The design of the century*

$$\sigma : \begin{cases} A_i \mapsto A_{i+5 \ (\mathrm{mod}\ 55)}, \\ B_j \mapsto B_{j+4 \ (\mathrm{mod}\ 44)}, j = 0, 1, \ldots, 43, \\ B_{44} \mapsto B_{44}. \end{cases}$$

| | | |
|---|---|---|
| B0 B1 B9 A0 | B4 B6 B23 A0 | B8 B11 B13 A0 |
| B12 B20 B10 A0 | B36 B5 B7 A0 | B2 B3 B38 A0 |
| B0 B4 B33 A1 | B40 B3 B6 A1 | B8 B24 B5 A1 |
| B16 B7 B11 A1 | B1 B2 B21 A1 | B9 B13 B42 A1 |
| B0 B6 B24 A2 | B8 B19 B40 A2 | B4 B31 B3 A2 |
| B9 B14 B43 A2 | B1 B7 B29 A2 | B5 B26 B2 A2 |
| B0 B14 B21 A3 | B4 B35 B41 A3 | B1 B10 B31 A3 |
| B5 B23 B44 A3 | B2 B7 B18 A3 | B22 B34 B3 A3 |
| B28 B1 B14 A4 | B4 B22 B26 A4 | B0 B38 B44 A4 |
| B29 B39 B2 A4 | B37 B5 B19 A4 | B3 B11 B35 A4 |
| A25 A29 A19 B0 | A20 A32 A5 B0 | A35 A48 A18 B0 |
| A15 A39 A11 B0 | A41 A43 A8 B0 | A21 A42 A13 B0 |
| A31 A14 A28 B0 | A16 A9 A12 B0 | A17 A23 A49 B0 |
| A37 A44 A33 B0 | A22 A34 A38 B0 | |
| A35 A36 A13 B1 | A10 A17 A18 B1 | A25 A44 A11 B1 |
| A20 A43 A19 B1 | A5 A37 A39 B1 | A15 A6 A12 B1 |
| A30 A23 A28 B1 | A26 A29 A34 B1 | A21 A38 A48 B1 |
| A27 A42 A9 B1 | A32 A49 A7 B1 | |
| A5 A7 A21 B2 | A20 A25 A46 B2 | A35 A41 A11 B2 |
| A40 A54 A15 B2 | A30 A47 A19 B2 | A36 A37 A23 B2 |
| A26 A39 A24 B2 | A16 A32 A53 B2 | A31 A12 A22 B2 |
| A17 A8 A9 B2 | A13 A28 A49 B2 | |
| A40 A43 A32 B3 | A35 A44 A15 B3 | A10 A20 A38 B3 |
| A5 A27 A47 B3 | A25 A6 A13 B3 | A21 A26 A36 B3 |
| A31 A42 A11 B3 | A16 A28 A34 B3 | A41 A9 A29 B3 |
| A12 A17 A48 B3 | A8 A24 A54 B3 | |
| A0 A11 A37 B44 | | |

Table 11.5: $S(2,4,109)$

$$\tau : \begin{cases} A_i \mapsto A_{i+3 \pmod{63}}, \ i = 0,1,\ldots,62, \\ B_j \mapsto B_{j+2 \pmod{42}}, \ j = 0,1,\ldots,41, \\ B_{42} \mapsto B_{43} \mapsto B_{44} \mapsto B_{42}, \\ C_0 \mapsto C_0. \end{cases}$$

| | | |
|---|---|---|
| $B0$ $B1$ $B9$ $A0$ | $B2$ $B4$ $B33$ $A0$ | $B34$ $B3$ $B5$ $A0$ |
| $B32$ $B7$ $B10$ $A0$ | $B6$ $B11$ $B44$ $A0$ | |
| $B0$ $B3$ $B4$ $A1$ | $B2$ $B8$ $B29$ $A1$ | $B6$ $B13$ $B22$ $A1$ |
| $B12$ $B37$ $B7$ $A1$ | $B1$ $B5$ $B23$ $A1$ | |
| $B0$ $B8$ $B18$ $A2$ | $B4$ $B16$ $B39$ $A2$ | $B1$ $B7$ $B17$ $A2$ |
| $B30$ $B3$ $B42$ $A2$ | $B2$ $B21$ $B44$ $A2$ | |
| $A39$ $A44$ $A26$ $B0$ | $A9$ $A18$ $A34$ $B0$ | $A6$ $A16$ $A28$ $B0$ |
| $A3$ $A14$ $A29$ $B0$ | $A33$ $A45$ $A10$ $B0$ | $A24$ $A42$ $A17$ $B0$ |
| $A30$ $A49$ $A19$ $B0$ | $A51$ $A8$ $A37$ $B0$ | $A36$ $A5$ $A32$ $B0$ |
| $A27$ $A7$ $A11$ $B0$ | $A21$ $A4$ $A13$ $B0$ | $A43$ $A50$ $A22$ $B0$ |
| $A25$ $A40$ $A23$ $B0$ | $A47$ $A56$ $A35$ $B0$ | |
| $A33$ $A34$ $A37$ $B1$ | $A42$ $A44$ $A27$ $B1$ | $A21$ $A24$ $A19$ $B1$ |
| $A6$ $A12$ $A45$ $B1$ | $A9$ $A16$ $A32$ $B1$ | $A3$ $A11$ $A30$ $B1$ |
| $A39$ $A5$ $A38$ $B1$ | $A36$ $A4$ $A26$ $B1$ | $A18$ $A53$ $A59$ $B1$ |
| $A43$ $A49$ $A29$ $B1$ | $A40$ $A50$ $A13$ $B1$ | $A7$ $A20$ $A52$ $B1$ |
| $A28$ $A47$ $A23$ $B1$ | $A25$ $A14$ $A17$ $B1$ | |
| $A0$ $A13$ $A42$ $B42$ | $A7$ $A8$ $A46$ $B42$ | |
| $B0$ $B14$ $B28$ $C0$ | $B1$ $B15$ $B29$ $C0$ | $B42$ $B43$ $B44$ $C0$ |
| $A0$ $A14$ $A37$ $C0$ | | |

Table 11.6: $S(2,4,184)$

$$\phi : \begin{cases} A_i \mapsto A_{i+2 \pmod{114}},\ i = 0,1,\ldots,113, \\ A_{114} \mapsto A_{114}, A_{115} \mapsto A_{115}, A_{116} \mapsto A_{116}, \\ B_j \mapsto B_{j+1 \pmod{57}},\ j = 0,1,\ldots,56, \\ C_0 \mapsto C_1 \mapsto C_2 \mapsto C_0, C_3 \mapsto C_4 \mapsto C_5 \mapsto C_3, C_6 \mapsto C_7 \mapsto C_8 \mapsto C_6, \\ D_0 \mapsto D_0. \end{cases}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B0 | B3 | B21 | A0 | B1 | B7 | B32 | A0 | B2 | B11 | B44 | A0 |
| B0 | B11 | B28 | A1 | B1 | B13 | B36 | A1 | B2 | B16 | B43 | A1 |
| C0 | C3 | C6 | A114 | C0 | C5 | C7 | A115 | C0 | C4 | C8 | A116 |
| A28 | A30 | A94 | B0 | A80 | A85 | A74 | B0 | A46 | A53 | A71 | B0 |
| A58 | A66 | A84 | B0 | A68 | A77 | A19 | B0 | A60 | A70 | A37 | B0 |
| A90 | A102 | A32 | B0 | A106 | A8 | A25 | B0 | A104 | A9 | A12 | B0 |
| A62 | A82 | A17 | B0 | A48 | A69 | A21 | B0 | A24 | A47 | A56 | B0 |
| A52 | A76 | A22 | B0 | A36 | A63 | A98 | B0 | A6 | A34 | A105 | B0 |
| A20 | A49 | A54 | B0 | A10 | A41 | A88 | B0 | A16 | A51 | A79 | B0 |
| A64 | A101 | A18 | B0 | A78 | A3 | A115 | B0 | A96 | A23 | A57 | B0 |
| A86 | A14 | A65 | B0 | A42 | A87 | A95 | B0 | A40 | A97 | A33 | B0 |
| A44 | A103 | A15 | B0 | A38 | A99 | A27 | B0 | A4 | A81 | A114 | B0 |
| A2 | A91 | A116 | B0 | A11 | A13 | A73 | B0 | A39 | A45 | A61 | B0 |
| A109 | A5 | A35 | B0 | A55 | A67 | A31 | B0 | A107 | A7 | A75 | B0 |
| B0 | B1 | B5 | C0 | A0 | A1 | A95 | C0 | A2 | A15 | A16 | C0 |
| B0 | B2 | B10 | C3 | A0 | A3 | A73 | C3 | A2 | A17 | A76 | C3 |
| B0 | B7 | B20 | C6 | A0 | A4 | A47 | C6 | A2 | A99 | A103 | C6 |
| C0 | C1 | C2 | D0 | C3 | C4 | C5 | D0 | C6 | C7 | C8 | D0 |
| B0 | B19 | B38 | D0 | A0 | A38 | A76 | D0 | A1 | A39 | A77 | D0 |
| A114 | A115 | A116 | D0 | | | | | | | | |

# Appendix A

# STS configurations of six or fewer blocks

This is a complete listing of all configurations of six or fewer blocks that can occur in Steiner triple systems.

The first column gives the blocks of configuration $X$ with its canonical labelling. The second column is $p(X)$, the number of points. The third column indicates the degrees of the points in the order 0, 1, ..., $p(X)-1$; generating configurations are starred. The fourth column gives order of the full automorphism group. The fifth column gives $\mu(X)$, as defined in section 8.3.1 (Definition 8.3.1).

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 | 3 | 111 | 6 | $\infty$ | $A_0$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 | 5 | 21111 | 8 | $\infty$ | $A_2$ |
| 012 345 | 6 | 111111 | 72 | 15 | $A_1$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 | 6 | 221211 | 6 | $\infty$ | $B_5$ |
| 012 034 056 | 7 | 3111111 | 48 | $\infty$ | $B_3$ |
| 012 034 156 | 7 | 2211111 | 8 | 15 | $B_4$ |
| 012 034 567 | 8 | 21111111 | 48 | 15 | $B_2$ |
| 012 345 678 | 9 | 111111111 | 1296 | 15 | $B_1$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 245 | 6 | 222222* | 24 | $\infty$ | $C_{16}$ |
| 012 034 135 236 | 7 | 2223111 | 6 | $\infty$ | $C_{15}$ |
| 012 034 135 246 | 7 | 2222211 | 4 | 14 | $C_{14}$ |
| 012 034 135 067 | 8 | 32121111 | 4 | 15 | $C_{11}$ |
| 012 034 135 267 | 8 | 22221111 | 4 | 15 | $C_{12}$ |
| 012 034 156 357 | 8 | 22121211 | 8 | 15 | $C_{10}$ |
| 012 034 056 078 | 9 | 411111111 | 384 | $\infty$ | $C_7$ |
| 012 034 056 178 | 9 | 321111111 | 16 | 15 | $C_8$ |
| 012 034 135 678 | 9 | 221211111 | 36 | 15 | $C_6$ |
| 012 034 156 278 | 9 | 222111111 | 48 | 15 | $C_{13}$ |
| 012 034 156 378 | 9 | 221211111 | 8 | 15 | $C_9$ |
| 012 034 056 789 | 10 | 3111111111 | 288 | 15 | $C_4$ |
| 012 034 156 789 | 10 | 2211111111 | 48 | 15 | $C_5$ |
| 012 034 567 589 | 10 | 2111121111 | 128 | 15 | $C_3$ |
| 012 034 567 89a | 11 | 21111111111 | 576 | 15 | $C_2$ |
| 012 345 678 9ab | 12 | 111111111111 | 31104 | 15 | $C_1$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 236 456 | 7 | 2223222* | 12 | 13 | $D_1$ |
| 012 034 135 236 146 | 7 | 2323212 | 8 | $\infty$ | $D_2$ |
| 012 034 135 236 147 | 8 | 23232111 | 2 | 14 | $D_3$ |
| 012 034 135 236 457 | 8 | 22232211 | 2 | 14 | $D_4$ |
| 012 034 135 245 067 | 8 | 32222211 | 8 | 15 | $D_5$ |
| 012 034 135 246 257 | 8 | 22322211 | 2 | 14 | $D_6$ |
| 012 034 135 246 567 | 8 | 22222221 | 4 | 14 | $D_7$ |
| 012 034 135 067 168 | 9 | 331211211 | 4 | 15 | $D_8$ |
| 012 034 135 067 268 | 9 | 322211211 | 2 | 15 | $D_9$ |
| 012 034 135 067 568 | 9 | 321212211 | 2 | 15 | $D_{10}$ |
| 012 034 135 236 078 | 9 | 322311111 | 4 | 15 | $D_{11}$ |
| 012 034 135 236 378 | 9 | 222411111 | 12 | 15 | $D_{12}$ |
| 012 034 135 236 478 | 9 | 222321111 | 4 | 15 | $D_{13}$ |
| 012 034 135 245 678 | 9 | 222222111 | 144 | 15 | $D_{14}$ |
| 012 034 135 246 078 | 9 | 322221111 | 8 | 14 | $D_{15}$ |
| 012 034 135 246 178 | 9 | 232221111 | 2 | 14 | $D_{16}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 246 578 | 9 | 222222111 | 4 | 14 | $D_{17}$ |
| 012 034 135 267 468 | 9 | 222221211 | 2 | 15 | $D_{18}$ |
| 012 034 156 357 468 | 9 | 221222211 | 12 | 15 | $D_{19}$ |
| 012 034 056 178 379 | 10 | 3212111211 | 4 | 15 | $D_{30}$ |
| 012 034 135 067 089 | 10 | 4212111111 | 16 | 15 | $D_{20}$ |
| 012 034 135 067 189 | 10 | 3312111111 | 8 | 15 | $D_{21}$ |
| 012 034 135 067 289 | 10 | 3222111111 | 4 | 15 | $D_{22}$ |
| 012 034 135 067 589 | 10 | 3212121111 | 8 | 15 | $D_{23}$ |
| 012 034 135 067 689 | 10 | 3212112111 | 4 | 15 | $D_{24}$ |
| 012 034 135 236 789 | 10 | 2223111111 | 36 | 15 | $D_{25}$ |
| 012 034 135 246 789 | 10 | 2222211111 | 24 | 14 | $D_{26}$ |
| 012 034 135 267 289 | 10 | 2232111111 | 16 | 15 | $D_{27}$ |
| 012 034 135 267 489 | 10 | 2222211111 | 8 | 15 | $D_{28}$ |
| 012 034 135 267 689 | 10 | 2222112111 | 4 | 15 | $D_{29}$ |
| 012 034 156 357 289 | 10 | 2222121111 | 4 | 15 | $D_{31}$ |
| 012 034 156 378 579 | 10 | 2212121211 | 10 | 15 | $D_{32}$ |
| 012 034 056 078 09a | 11 | 51111111111 | 3840 | $\infty$ | $D_{33}$ |
| 012 034 056 078 19a | 11 | 42111111111 | 96 | 15 | $D_{34}$ |
| 012 034 056 178 19a | 11 | 33111111111 | 128 | 15 | $D_{35}$ |
| 012 034 056 178 29a | 11 | 32211111111 | 64 | 15 | $D_{36}$ |
| 012 034 056 178 39a | 11 | 32121111111 | 16 | 15 | $D_{37}$ |
| 012 034 056 178 79a | 11 | 32111112111 | 16 | 15 | $D_{38}$ |
| 012 034 135 067 89a | 11 | 32121111111 | 24 | 15 | $D_{39}$ |
| 012 034 135 267 89a | 11 | 22221111111 | 24 | 15 | $D_{40}$ |
| 012 034 135 678 69a | 11 | 22121121111 | 48 | 15 | $D_{41}$ |
| 012 034 156 278 39a | 11 | 22221111111 | 16 | 15 | $D_{42}$ |
| 012 034 156 357 89a | 11 | 22121211111 | 48 | 15 | $D_{43}$ |
| 012 034 156 378 59a | 11 | 22121211111 | 8 | 15 | $D_{44}$ |
| 012 034 056 078 9ab | 12 | 411111111111 | 2304 | 15 | $D_{45}$ |
| 012 034 056 178 9ab | 12 | 321111111111 | 96 | 15 | $D_{46}$ |
| 012 034 056 789 7ab | 12 | 311111121111 | 384 | 15 | $D_{47}$ |
| 012 034 135 678 9ab | 12 | 221211111111 | 432 | 15 | $D_{48}$ |
| 012 034 156 278 9ab | 12 | 222111111111 | 288 | 15 | $D_{49}$ |
| 012 034 156 378 9ab | 12 | 221211111111 | 48 | 15 | $D_{50}$ |
| 012 034 156 789 7ab | 12 | 221111121111 | 64 | 15 | $D_{51}$ |
| 012 034 056 789 abc | 13 | 3111111111111 | 3456 | 15 | $D_{52}$ |
| 012 034 156 789 abc | 13 | 2211111111111 | 576 | 15 | $D_{53}$ |
| 012 034 567 589 abc | 13 | 2111121111111 | 768 | 15 | $D_{54}$ |
| 012 034 567 89a bcd | 14 | 21111111111111 | 10368 | 15 | $D_{55}$ |
| 012 345 678 9ab cde | 15 | 111111111111111 | 933120 | 15 | $D_{56}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 236 146 245 | 7 | 2333322* | 24 | $\infty$ | $E_1$ |
| 012 034 135 236 147 567 | 8 | 23232222* | 2 | 13 | $E_2$ |
| 012 034 135 246 257 367 | 8 | 22332222* | 12 | 13 | $E_3$ |
| 012 034 135 267 468 578 | 9 | 222222222* | 12 | 15 | $E_4$ |
| 012 034 156 357 468 278 | 9 | 222222222* | 72 | 15 | $E_5$ |
| 012 034 135 236 146 057 | 8 | 33232221 | 2 | 14 | $E_6$ |
| 012 034 135 236 146 247 | 8 | 23333121 | 8 | 13 | $E_7$ |
| 012 034 135 236 147 257 | 8 | 23332212 | 2 | 13 | $E_8$ |
| 012 034 135 236 146 078 | 9 | 332321211 | 4 | 15 | $E_9$ |
| 012 034 135 236 146 178 | 9 | 242321211 | 8 | 15 | $E_{10}$ |
| 012 034 135 236 146 578 | 9 | 232322211 | 16 | 15 | $E_{11}$ |
| 012 034 135 236 147 058 | 9 | 332322111 | 6 | 14 | $E_{12}$ |
| 012 034 135 236 147 168 | 9 | 242321211 | 2 | 14 | $E_{13}$ |
| 012 034 135 236 147 248 | 9 | 233331111 | 8 | 14 | $E_{14}$ |
| 012 034 135 236 147 258 | 9 | 233322111 | 1 | 14 | $E_{15}$ |
| 012 034 135 236 147 468 | 9 | 232331211 | 1 | 14 | $E_{16}$ |
| 012 034 135 236 147 568 | 9 | 232322211 | 1 | 14 | $E_{17}$ |
| 012 034 135 236 147 678 | 9 | 232321221 | 2 | 14 | $E_{18}$ |
| 012 034 135 236 456 078 | 9 | 322322211 | 4 | 13 | $E_{19}$ |
| 012 034 135 236 456 378 | 9 | 222422211 | 24 | 13 | $E_{20}$ |
| 012 034 135 236 457 078 | 9 | 322322121 | 1 | 14 | $E_{21}$ |
| 012 034 135 236 457 278 | 9 | 223322121 | 2 | 14 | $E_{22}$ |
| 012 034 135 236 457 378 | 9 | 222422121 | 4 | 14 | $E_{23}$ |
| 012 034 135 236 457 468 | 9 | 222332211 | 2 | 14 | $E_{24}$ |
| 012 034 135 236 457 678 | 9 | 222322221 | 2 | 14 | $E_{25}$ |
| 012 034 135 245 067 168 | 9 | 332222211 | 2 | 15 | $E_{26}$ |
| 012 034 135 245 067 568 | 9 | 322223211 | 8 | 15 | $E_{27}$ |
| 012 034 135 246 257 168 | 9 | 233222211 | 2 | 14 | $E_{28}$ |
| 012 034 135 246 257 368 | 9 | 223322211 | 2 | 14 | $E_{29}$ |
| 012 034 135 246 257 458 | 9 | 223233111 | 6 | 14 | $E_{30}$ |
| 012 034 135 246 257 568 | 9 | 223223211 | 1 | 14 | $E_{31}$ |
| 012 034 135 246 257 678 | 9 | 223222221 | 2 | 14 | $E_{32}$ |
| 012 034 135 246 567 078 | 9 | 322222221 | 4 | 14 | $E_{33}$ |
| 012 034 135 246 567 178 | 9 | 232222221 | 1 | 14 | $E_{34}$ |
| 012 034 135 067 168 369 | 10 | 3313113111 | 24 | 15 | $E_{35}$ |
| 012 034 135 067 168 469 | 10 | 3312213111 | 2 | 15 | $E_{36}$ |
| 012 034 135 067 168 479 | 10 | 3312212211 | 2 | 15 | $E_{37}$ |
| 012 034 135 067 168 579 | 10 | 3312122211 | 2 | 15 | $E_{38}$ |
| 012 034 135 067 268 479 | 10 | 3222212211 | 6 | 15 | $E_{39}$ |
| 012 034 135 067 268 569 | 10 | 3222123111 | 1 | 15 | $E_{40}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 067 268 579 | 10 | 3222122211 | 1 | 15 | $E_{41}$ |
| 012 034 135 067 268 589 | 10 | 3222122121 | 2 | 15 | $E_{42}$ |
| 012 034 135 067 568 089 | 10 | 4212122121 | 8 | 15 | $E_{43}$ |
| 012 034 135 067 568 289 | 10 | 3222122121 | 1 | 15 | $E_{44}$ |
| 012 034 135 067 568 579 | 10 | 3212132211 | 8 | 15 | $E_{45}$ |
| 012 034 135 067 568 789 | 10 | 3212122221 | 2 | 15 | $E_{46}$ |
| 012 034 135 236 078 179 | 10 | 3323111211 | 2 | 15 | $E_{47}$ |
| 012 034 135 236 078 379 | 10 | 3224111211 | 2 | 15 | $E_{48}$ |
| 012 034 135 236 078 479 | 10 | 3223211211 | 2 | 15 | $E_{49}$ |
| 012 034 135 236 078 579 | 10 | 3223121211 | 1 | 15 | $E_{50}$ |
| 012 034 135 236 146 789 | 10 | 2323212111 | 48 | 15 | $E_{51}$ |
| 012 034 135 236 147 089 | 10 | 3323211111 | 4 | 14 | $E_{52}$ |
| 012 034 135 236 147 189 | 10 | 2423211111 | 2 | 14 | $E_{53}$ |
| 012 034 135 236 147 289 | 10 | 2333211111 | 2 | 14 | $E_{54}$ |
| 012 034 135 236 147 589 | 10 | 2323221111 | 4 | 14 | $E_{55}$ |
| 012 034 135 236 147 689 | 10 | 2323212111 | 2 | 14 | $E_{56}$ |
| 012 034 135 236 378 479 | 10 | 2224211211 | 2 | 15 | $E_{57}$ |
| 012 034 135 236 456 789 | 10 | 2223222111 | 72 | 13 | $E_{58}$ |
| 012 034 135 236 457 089 | 10 | 3223221111 | 2 | 14 | $E_{59}$ |
| 012 034 135 236 457 289 | 10 | 2233221111 | 4 | 14 | $E_{60}$ |
| 012 034 135 236 457 389 | 10 | 2224221111 | 4 | 14 | $E_{61}$ |
| 012 034 135 236 457 489 | 10 | 2223321111 | 2 | 14 | $E_{62}$ |
| 012 034 135 236 457 689 | 10 | 2223222111 | 4 | 14 | $E_{63}$ |
| 012 034 135 236 457 789 | 10 | 2223221211 | 4 | 14 | $E_{64}$ |
| 012 034 135 236 478 579 | 10 | 2223221211 | 2 | 15 | $E_{65}$ |
| 012 034 135 245 067 089 | 10 | 4222221111 | 32 | 15 | $E_{66}$ |
| 012 034 135 245 067 189 | 10 | 3322221111 | 8 | 15 | $E_{67}$ |
| 012 034 135 245 067 589 | 10 | 3222231111 | 32 | 15 | $E_{68}$ |
| 012 034 135 245 067 689 | 10 | 3222222111 | 8 | 15 | $E_{69}$ |
| 012 034 135 246 078 179 | 10 | 3322211211 | 1 | 14 | $E_{70}$ |
| 012 034 135 246 078 579 | 10 | 3222221211 | 2 | 14 | $E_{71}$ |
| 012 034 135 246 178 279 | 10 | 2332211211 | 2 | 14 | $E_{72}$ |
| 012 034 135 246 178 379 | 10 | 2323211211 | 2 | 14 | $E_{73}$ |
| 012 034 135 246 178 479 | 10 | 2322311211 | 2 | 14 | $E_{74}$ |
| 012 034 135 246 178 579 | 10 | 2322221211 | 1 | 14 | $E_{75}$ |
| 012 034 135 246 178 679 | 10 | 2322212211 | 1 | 14 | $E_{76}$ |
| 012 034 135 246 257 089 | 10 | 3232221111 | 2 | 14 | $E_{77}$ |
| 012 034 135 246 257 289 | 10 | 2242221111 | 4 | 14 | $E_{78}$ |
| 012 034 135 246 257 389 | 10 | 2233221111 | 4 | 14 | $E_{79}$ |
| 012 034 135 246 257 489 | 10 | 2232321111 | 2 | 14 | $E_{80}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 246 257 689 | 10 | 2232222111 | 2 | 14 | $E_{81}$ |
| 012 034 135 246 567 089 | 10 | 3222222111 | 8 | 14 | $E_{82}$ |
| 012 034 135 246 567 189 | 10 | 2322222111 | 2 | 14 | $E_{83}$ |
| 012 034 135 246 567 589 | 10 | 2222232111 | 4 | 14 | $E_{84}$ |
| 012 034 135 246 567 789 | 10 | 2222222211 | 8 | 14 | $E_{85}$ |
| 012 034 135 246 578 679 | 10 | 2222222211 | 4 | 14 | $E_{86}$ |
| 012 034 135 267 468 569 | 10 | 2222223111 | 6 | 15 | $E_{87}$ |
| 012 034 135 267 468 579 | 10 | 2222222211 | 2 | 15 | $E_{88}$ |
| 012 034 135 267 468 789 | 10 | 2222212221 | 4 | 15 | $E_{89}$ |
| 012 034 156 357 468 279 | 10 | 2222222211 | 8 | 15 | $E_{90}$ |
| 012 034 056 178 379 48a | 11 | 32122112211 | 4 | 15 | $E_{91}$ |
| 012 034 056 178 379 57a | 11 | 32121213111 | 12 | 15 | $E_{92}$ |
| 012 034 056 178 379 58a | 11 | 32121212211 | 2 | 15 | $E_{93}$ |
| 012 034 135 067 089 68a | 11 | 42121121211 | 8 | 15 | $E_{94}$ |
| 012 034 135 067 168 09a | 11 | 43121121111 | 4 | 15 | $E_{95}$ |
| 012 034 135 067 168 29a | 11 | 33221121111 | 8 | 15 | $E_{96}$ |
| 012 034 135 067 168 39a | 11 | 33131121111 | 4 | 15 | $E_{97}$ |
| 012 034 135 067 168 49a | 11 | 33122121111 | 2 | 15 | $E_{98}$ |
| 012 034 135 067 189 68a | 11 | 33121121211 | 2 | 15 | $E_{99}$ |
| 012 034 135 067 268 09a | 11 | 42221121111 | 4 | 15 | $E_{100}$ |
| 012 034 135 067 268 19a | 11 | 33221121111 | 2 | 15 | $E_{101}$ |
| 012 034 135 067 268 39a | 11 | 32231121111 | 2 | 15 | $E_{102}$ |
| 012 034 135 067 268 49a | 11 | 32222121111 | 2 | 15 | $E_{103}$ |
| 012 034 135 067 268 59a | 11 | 32221221111 | 2 | 15 | $E_{104}$ |
| 012 034 135 067 289 48a | 11 | 32222111211 | 4 | 15 | $E_{105}$ |
| 012 034 135 067 289 58a | 11 | 32221211211 | 2 | 15 | $E_{106}$ |
| 012 034 135 067 289 68a | 11 | 32221121211 | 1 | 15 | $E_{107}$ |
| 012 034 135 067 568 09a | 11 | 42121221111 | 4 | 15 | $E_{108}$ |
| 012 034 135 067 568 19a | 11 | 33121221111 | 2 | 15 | $E_{109}$ |
| 012 034 135 067 568 29a | 11 | 32221221111 | 2 | 15 | $E_{110}$ |
| 012 034 135 067 568 59a | 11 | 32121321111 | 4 | 15 | $E_{111}$ |
| 012 034 135 067 568 69a | 11 | 32121231111 | 4 | 15 | $E_{112}$ |
| 012 034 135 067 568 79a | 11 | 32121222111 | 4 | 15 | $E_{113}$ |
| 012 034 135 067 568 89a | 11 | 32121221211 | 4 | 15 | $E_{114}$ |
| 012 034 135 067 589 68a | 11 | 32121221211 | 2 | 15 | $E_{115}$ |
| 012 034 135 067 689 78a | 11 | 32121122211 | 4 | 15 | $E_{116}$ |
| 012 034 135 236 078 09a | 11 | 42231111111 | 16 | 15 | $E_{117}$ |
| 012 034 135 236 078 19a | 11 | 33231111111 | 8 | 15 | $E_{118}$ |
| 012 034 135 236 078 39a | 11 | 32241111111 | 8 | 15 | $E_{119}$ |
| 012 034 135 236 078 49a | 11 | 32232111111 | 8 | 15 | $E_{120}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 236 078 59a | 11 | 32231211111 | 4 | 15 | $E_{121}$ |
| 012 034 135 236 078 79a | 11 | 32231112111 | 4 | 15 | $E_{122}$ |
| 012 034 135 236 147 89a | 11 | 23232111111 | 12 | 14 | $E_{123}$ |
| 012 034 135 236 378 39a | 11 | 22251111111 | 48 | 15 | $E_{124}$ |
| 012 034 135 236 378 49a | 11 | 22242111111 | 8 | 15 | $E_{125}$ |
| 012 034 135 236 378 79a | 11 | 22241112111 | 12 | 15 | $E_{126}$ |
| 012 034 135 236 457 89a | 11 | 22232211111 | 12 | 14 | $E_{127}$ |
| 012 034 135 236 478 49a | 11 | 22233111111 | 16 | 15 | $E_{128}$ |
| 012 034 135 236 478 59a | 11 | 22232211111 | 8 | 15 | $E_{129}$ |
| 012 034 135 236 478 79a | 11 | 22232112111 | 4 | 15 | $E_{130}$ |
| 012 034 135 245 067 89a | 11 | 32222211111 | 48 | 15 | $E_{131}$ |
| 012 034 135 245 678 69a | 11 | 22222221111 | 192 | 15 | $E_{132}$ |
| 012 034 135 246 078 09a | 11 | 42222111111 | 32 | 14 | $E_{133}$ |
| 012 034 135 246 078 19a | 11 | 33222111111 | 4 | 14 | $E_{134}$ |
| 012 034 135 246 078 59a | 11 | 32222211111 | 8 | 14 | $E_{135}$ |
| 012 034 135 246 078 79a | 11 | 32222112111 | 8 | 14 | $E_{136}$ |
| 012 034 135 246 178 19a | 11 | 24222111111 | 8 | 14 | $E_{137}$ |
| 012 034 135 246 178 29a | 11 | 23322111111 | 8 | 14 | $E_{138}$ |
| 012 034 135 246 178 39a | 11 | 23232111111 | 8 | 14 | $E_{139}$ |
| 012 034 135 246 178 49a | 11 | 23223111111 | 8 | 14 | $E_{140}$ |
| 012 034 135 246 178 59a | 11 | 23222211111 | 4 | 14 | $E_{141}$ |
| 012 034 135 246 178 69a | 11 | 23222121111 | 4 | 14 | $E_{142}$ |
| 012 034 135 246 178 79a | 11 | 23222112111 | 2 | 14 | $E_{143}$ |
| 012 034 135 246 257 89a | 11 | 22322211111 | 12 | 14 | $E_{144}$ |
| 012 034 135 246 567 89a | 11 | 22222221111 | 24 | 14 | $E_{145}$ |
| 012 034 135 246 578 59a | 11 | 22222311111 | 16 | 14 | $E_{146}$ |
| 012 034 135 246 578 69a | 11 | 22222221111 | 16 | 14 | $E_{147}$ |
| 012 034 135 246 578 79a | 11 | 22222212111 | 4 | 14 | $E_{148}$ |
| 012 034 135 267 468 29a | 11 | 22322121111 | 2 | 15 | $E_{149}$ |
| 012 034 135 267 468 59a | 11 | 22222221111 | 4 | 15 | $E_{150}$ |
| 012 034 135 267 468 69a | 11 | 22222131111 | 4 | 15 | $E_{151}$ |
| 012 034 135 267 468 79a | 11 | 22222122111 | 2 | 15 | $E_{152}$ |
| 012 034 135 267 489 68a | 11 | 22222121211 | 2 | 15 | $E_{153}$ |
| 012 034 135 267 689 78a | 11 | 22221122211 | 8 | 15 | $E_{154}$ |
| 012 034 156 357 289 48a | 11 | 22222211211 | 4 | 15 | $E_{155}$ |
| 012 034 156 357 289 78a | 11 | 22221212211 | 4 | 15 | $E_{156}$ |
| 012 034 156 357 468 29a | 11 | 22222221111 | 8 | 15 | $E_{157}$ |
| 012 034 056 078 19a 39b | 12 | 421211111211 | 16 | 15 | $E_{158}$ |
| 012 034 056 178 29a 79b | 12 | 322111121211 | 16 | 15 | $E_{159}$ |
| 012 034 056 178 379 1ab | 12 | 331211121111 | 8 | 15 | $E_{160}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 056 178 379 2ab | 12 | 322211121111 | 4 | 15 | $E_{161}$ |
| 012 034 056 178 379 5ab | 12 | 321212121111 | 4 | 15 | $E_{162}$ |
| 012 034 056 178 379 7ab | 12 | 321211131111 | 16 | 15 | $E_{163}$ |
| 012 034 056 178 379 8ab | 12 | 321211122111 | 4 | 15 | $E_{164}$ |
| 012 034 056 178 39a 79b | 12 | 321211121211 | 4 | 15 | $E_{165}$ |
| 012 034 135 067 089 0ab | 12 | 521211111111 | 96 | 15 | $E_{166}$ |
| 012 034 135 067 089 1ab | 12 | 431211111111 | 16 | 15 | $E_{167}$ |
| 012 034 135 067 089 2ab | 12 | 422211111111 | 16 | 15 | $E_{168}$ |
| 012 034 135 067 089 5ab | 12 | 421212111111 | 32 | 15 | $E_{169}$ |
| 012 034 135 067 089 6ab | 12 | 421211211111 | 8 | 15 | $E_{170}$ |
| 012 034 135 067 168 9ab | 12 | 331211211111 | 24 | 15 | $E_{171}$ |
| 012 034 135 067 189 2ab | 12 | 332211111111 | 16 | 15 | $E_{172}$ |
| 012 034 135 067 189 3ab | 12 | 331311111111 | 48 | 15 | $E_{173}$ |
| 012 034 135 067 189 4ab | 12 | 331221111111 | 8 | 15 | $E_{174}$ |
| 012 034 135 067 189 6ab | 12 | 331211211111 | 4 | 15 | $E_{175}$ |
| 012 034 135 067 268 9ab | 12 | 322211211111 | 12 | 15 | $E_{176}$ |
| 012 034 135 067 289 2ab | 12 | 323211111111 | 16 | 15 | $E_{177}$ |
| 012 034 135 067 289 4ab | 12 | 322221111111 | 16 | 15 | $E_{178}$ |
| 012 034 135 067 289 5ab | 12 | 322212111111 | 8 | 15 | $E_{179}$ |
| 012 034 135 067 289 6ab | 12 | 322211211111 | 4 | 15 | $E_{180}$ |
| 012 034 135 067 289 8ab | 12 | 322211112111 | 4 | 15 | $E_{181}$ |
| 012 034 135 067 568 9ab | 12 | 321212211111 | 12 | 15 | $E_{182}$ |
| 012 034 135 067 589 5ab | 12 | 321213111111 | 32 | 15 | $E_{183}$ |
| 012 034 135 067 589 6ab | 12 | 321212211111 | 8 | 15 | $E_{184}$ |
| 012 034 135 067 589 8ab | 12 | 321212112111 | 8 | 15 | $E_{185}$ |
| 012 034 135 067 689 6ab | 12 | 321211311111 | 16 | 15 | $E_{186}$ |
| 012 034 135 067 689 7ab | 12 | 321211221111 | 16 | 15 | $E_{187}$ |
| 012 034 135 067 689 8ab | 12 | 321211212111 | 4 | 15 | $E_{188}$ |
| 012 034 135 236 078 9ab | 12 | 322311111111 | 24 | 15 | $E_{189}$ |
| 012 034 135 236 378 9ab | 12 | 222411111111 | 72 | 15 | $E_{190}$ |
| 012 034 135 236 478 9ab | 12 | 222321111111 | 24 | 15 | $E_{191}$ |
| 012 034 135 236 789 7ab | 12 | 222311121111 | 48 | 15 | $E_{192}$ |
| 012 034 135 245 678 9ab | 12 | 222222111111 | 1728 | 15 | $E_{193}$ |
| 012 034 135 246 078 9ab | 12 | 322221111111 | 48 | 14 | $E_{194}$ |
| 012 034 135 246 178 9ab | 12 | 232221111111 | 12 | 14 | $E_{195}$ |
| 012 034 135 246 578 9ab | 12 | 222222111111 | 24 | 14 | $E_{196}$ |
| 012 034 135 246 789 7ab | 12 | 222221121111 | 32 | 14 | $E_{197}$ |
| 012 034 135 267 289 2ab | 12 | 224211111111 | 96 | 15 | $E_{198}$ |
| 012 034 135 267 289 4ab | 12 | 223221111111 | 16 | 15 | $E_{199}$ |
| 012 034 135 267 289 6ab | 12 | 223211211111 | 8 | 15 | $E_{200}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 267 468 9ab | 12 | 222221211111 | 12 | 15 | $E_{201}$ |
| 012 034 135 267 489 5ab | 12 | 222222111111 | 48 | 15 | $E_{202}$ |
| 012 034 135 267 489 6ab | 12 | 222221211111 | 4 | 15 | $E_{203}$ |
| 012 034 135 267 689 6ab | 12 | 222211311111 | 16 | 15 | $E_{204}$ |
| 012 034 135 267 689 7ab | 12 | 222211221111 | 16 | 15 | $E_{205}$ |
| 012 034 135 267 689 8ab | 12 | 222211212111 | 4 | 15 | $E_{206}$ |
| 012 034 135 678 69a 79b | 12 | 221211221211 | 72 | 15 | $E_{207}$ |
| 012 034 156 278 39a 59b | 12 | 222212111211 | 4 | 15 | $E_{208}$ |
| 012 034 156 357 289 4ab | 12 | 222222111111 | 8 | 15 | $E_{209}$ |
| 012 034 156 357 289 7ab | 12 | 222212121111 | 16 | 15 | $E_{210}$ |
| 012 034 156 357 289 8ab | 12 | 222212112111 | 4 | 15 | $E_{211}$ |
| 012 034 156 357 468 9ab | 12 | 221222211111 | 72 | 15 | $E_{212}$ |
| 012 034 156 378 59a 79b | 12 | 221212121211 | 12 | 15 | $E_{213}$ |
| 012 034 056 078 09a 0bc | 13 | 6111111111111 | 46080 | $\infty$ | $E_{214}$ |
| 012 034 056 078 09a 1bc | 13 | 5211111111111 | 768 | 15 | $E_{215}$ |
| 012 034 056 078 19a 1bc | 13 | 4311111111111 | 384 | 15 | $E_{216}$ |
| 012 034 056 078 19a 2bc | 13 | 4221111111111 | 384 | 15 | $E_{217}$ |
| 012 034 056 078 19a 3bc | 13 | 4212111111111 | 64 | 15 | $E_{218}$ |
| 012 034 056 078 19a 9bc | 13 | 4211111112111 | 96 | 15 | $E_{219}$ |
| 012 034 056 178 19a 2bc | 13 | 3321111111111 | 256 | 15 | $E_{220}$ |
| 012 034 056 178 19a 3bc | 13 | 3312111111111 | 32 | 15 | $E_{221}$ |
| 012 034 056 178 29a 3bc | 13 | 3222111111111 | 32 | 15 | $E_{222}$ |
| 012 034 056 178 29a 7bc | 13 | 3221111211111 | 32 | 15 | $E_{223}$ |
| 012 034 056 178 379 abc | 13 | 3212111211111 | 24 | 15 | $E_{224}$ |
| 012 034 056 178 39a 5bc | 13 | 3212121111111 | 48 | 15 | $E_{225}$ |
| 012 034 056 178 39a 7bc | 13 | 3212111211111 | 8 | 15 | $E_{226}$ |
| 012 034 056 178 79a 7bc | 13 | 3211111311111 | 128 | 15 | $E_{227}$ |
| 012 034 056 178 79a 8bc | 13 | 3211111221111 | 64 | 15 | $E_{228}$ |
| 012 034 056 178 79a 9bc | 13 | 3211111212111 | 16 | 15 | $E_{229}$ |
| 012 034 135 067 089 abc | 13 | 4212111111111 | 96 | 15 | $E_{230}$ |
| 012 034 135 067 189 abc | 13 | 3312111111111 | 48 | 15 | $E_{231}$ |
| 012 034 135 067 289 abc | 13 | 3222111111111 | 24 | 15 | $E_{232}$ |
| 012 034 135 067 589 abc | 13 | 3212121111111 | 48 | 15 | $E_{233}$ |
| 012 034 135 067 689 abc | 13 | 3212112111111 | 24 | 15 | $E_{234}$ |
| 012 034 135 067 89a 8bc | 13 | 3212111121111 | 32 | 15 | $E_{235}$ |
| 012 034 135 236 789 abc | 13 | 2223111111111 | 432 | 15 | $E_{236}$ |
| 012 034 135 246 789 abc | 13 | 2222211111111 | 288 | 14 | $E_{237}$ |
| 012 034 135 267 289 abc | 13 | 2232111111111 | 96 | 15 | $E_{238}$ |
| 012 034 135 267 489 abc | 13 | 2222211111111 | 48 | 15 | $E_{239}$ |
| 012 034 135 267 689 abc | 13 | 2222112111111 | 24 | 15 | $E_{240}$ |

| Blocks | $p(X)$ | degrees | \|Aut(X)\| | $\mu(X)$ | $X$ |
|---|---|---|---|---|---|
| 012 034 135 267 89a 8bc | 13 | 2222111121111 | 32 | 15 | $E_{241}$ |
| 012 034 135 678 69a 6bc | 13 | 2212113111111 | 288 | 15 | $E_{242}$ |
| 012 034 135 678 69a 7bc | 13 | 2212112211111 | 48 | 15 | $E_{243}$ |
| 012 034 156 278 39a 4bc | 13 | 2222211111111 | 128 | 15 | $E_{244}$ |
| 012 034 156 278 39a 5bc | 13 | 2222121111111 | 16 | 15 | $E_{245}$ |
| 012 034 156 278 39a 9bc | 13 | 2222111112111 | 16 | 15 | $E_{246}$ |
| 012 034 156 357 289 abc | 13 | 2222121111111 | 24 | 15 | $E_{247}$ |
| 012 034 156 357 89a 8bc | 13 | 2212121121111 | 64 | 15 | $E_{248}$ |
| 012 034 156 378 579 abc | 13 | 2212121211111 | 60 | 15 | $E_{249}$ |
| 012 034 156 378 59a 7bc | 13 | 2212121211111 | 8 | 15 | $E_{250}$ |
| 012 034 056 078 09a bcd | 14 | 51111111111111 | 23040 | 15 | $E_{251}$ |
| 012 034 056 078 19a bcd | 14 | 42111111111111 | 576 | 15 | $E_{252}$ |
| 012 034 056 078 9ab 9cd | 14 | 41111111121111 | 3072 | 15 | $E_{253}$ |
| 012 034 056 178 19a bcd | 14 | 33111111111111 | 768 | 15 | $E_{254}$ |
| 012 034 056 178 29a bcd | 14 | 32211111111111 | 384 | 15 | $E_{255}$ |
| 012 034 056 178 39a bcd | 14 | 32121111111111 | 96 | 15 | $E_{256}$ |
| 012 034 056 178 79a bcd | 14 | 32111112111111 | 96 | 15 | $E_{257}$ |
| 012 034 056 178 9ab 9cd | 14 | 32111111121111 | 128 | 15 | $E_{258}$ |
| 012 034 056 789 7ab 7cd | 14 | 31111113111111 | 4608 | 15 | $E_{259}$ |
| 012 034 056 789 7ab 8cd | 14 | 31111112211111 | 384 | 15 | $E_{260}$ |
| 012 034 135 067 89a bcd | 14 | 32121111111111 | 288 | 15 | $E_{261}$ |
| 012 034 135 267 89a bcd | 14 | 22221111111111 | 288 | 15 | $E_{262}$ |
| 012 034 135 678 69a bcd | 14 | 22121121111111 | 288 | 15 | $E_{263}$ |
| 012 034 156 278 39a bcd | 14 | 22221111111111 | 96 | 15 | $E_{264}$ |
| 012 034 156 278 9ab 9cd | 14 | 22211111121111 | 384 | 15 | $E_{265}$ |
| 012 034 156 357 89a bcd | 14 | 22121211111111 | 576 | 15 | $E_{266}$ |
| 012 034 156 378 59a bcd | 14 | 22121211111111 | 48 | 15 | $E_{267}$ |
| 012 034 156 378 9ab 9cd | 14 | 22121111121111 | 64 | 15 | $E_{268}$ |
| 012 034 156 789 7ab 8cd | 14 | 22111112211111 | 128 | 15 | $E_{269}$ |
| 012 034 056 078 9ab cde | 15 | 411111111111111 | 27648 | 15 | $E_{270}$ |
| 012 034 056 178 9ab cde | 15 | 321111111111111 | 1152 | 15 | $E_{271}$ |
| 012 034 056 789 7ab cde | 15 | 311111121111111 | 2304 | 15 | $E_{272}$ |
| 012 034 135 678 9ab cde | 15 | 221211111111111 | 7776 | 15 | $E_{273}$ |
| 012 034 156 278 9ab cde | 15 | 222111111111111 | 3456 | 15 | $E_{274}$ |
| 012 034 156 378 9ab cde | 15 | 221211111111111 | 576 | 15 | $E_{275}$ |
| 012 034 156 789 7ab cde | 15 | 221111121111111 | 384 | 15 | $E_{276}$ |
| 012 034 567 589 abc ade | 15 | 211112111121111 | 3072 | 15 | $E_{277}$ |
| 012 034 056 789 abc def | 16 | 3111111111111111 | 62208 | 15 | $E_{278}$ |
| 012 034 156 789 abc def | 16 | 2211111111111111 | 10368 | 15 | $E_{279}$ |
| 012 034 567 589 abc def | 16 | 2111121111111111 | 9216 | 15 | $E_{280}$ |
| 012 034 567 89a bcd efg | 17 | 21111111111111111 | 248832 | 15 | $E_{281}$ |
| 012 345 678 9ab cde fgh | 18 | 111111111111111111 | 33592320 | 15 | $E_{282}$ |

# Appendix B

# Formulae for 6-block STS configurations

The formulae for the number of occurrences of 6-block configurations in Steiner triple systems. There are seven generating configurations: $C_{16}$, $D_1$, $E_1$, $E_2$, $E_3$, $E_4$ and $E_5$. Formulae for $n$-block STS configurations are given in [34, 16] for $n \leq 4$ and [17] for $n = 5$.

We write $n_v$ for $v(v-1)(v-3)$.

$$\begin{aligned}
e_6 &= 12\,(c_{16} - e_1) \\
e_7 &= 3\,(c_{16} - e_1) \\
e_8 &= 6\,d_1 \\
e_9 &= 6\,(-9\,c_{16} + 2\,e_1 + c_{16}\,v) \\
e_{10} &= 3\,c_{16}\,(-7 + v) \\
e_{11} &= (3\,(-11\,c_{16} + 4\,e_1 + c_{16}\,v))/2 \\
e_{12} &= -12\,c_{16} + 8\,e_1 + n_v/6 \\
e_{13} &= (-24\,c_{16} + n_v)/2 \\
e_{14} &= -6\,c_{16} + 3\,e_1 + n_v/8 \\
e_{15} &= -48\,c_{16} - 12\,d_1 + 24\,e_1 + n_v \\
e_{16} &= -24\,c_{16} - 12\,d_1 + n_v \\
e_{17} &= -48\,c_{16} - 12\,d_1 + 24\,e_1 - 2\,e_2 + n_v \\
e_{18} &= -12\,c_{16} - 6\,d_1 - e_2 + n_v/2 \\
e_{19} &= 3\,d_1\,(-9 + v) \\
e_{20} &= (d_1\,(-7 + v))/2
\end{aligned}$$

$$
\begin{aligned}
e_{21} &= -24\,c_{16} - 12\,d_1 - 2\,e_2 + n_v \\
e_{22} &= -12\,c_{16} - 6\,d_1 + n_v/2 \\
e_{23} &= -6\,c_{16} - 3\,d_1 + n_v/4 \\
e_{24} &= -24\,c_{16} - 6\,d_1 + 12\,e_1 + n_v/2 \\
e_{25} &= -12\,c_{16} - 6\,d_1 - 2\,e_2 + n_v/2 \\
e_{26} &= 12\,(-8\,c_{16} + e_1 + c_{16}\,v) \\
e_{27} &= 3\,c_{16}\,(-7 + v) \\
e_{28} &= -12\,c_{16} - e_2 + n_v/2 \\
e_{29} &= -12\,c_{16} - 6\,e_3 + n_v/2 \\
e_{30} &= -4\,c_{16} + n_v/6 \\
e_{31} &= -24\,c_{16} - 12\,d_1 + n_v \\
e_{32} &= -12\,c_{16} - 2\,e_2 - 6\,e_3 + n_v/2 \\
e_{33} &= -6\,c_{16} - 3\,d_1 + n_v/4 \\
e_{34} &= -24\,c_{16} - 12\,d_1 - 2\,e_2 + n_v \\
e_{35} &= 3\,c_{16} - e_1 - (3\,n_v)/8 + (n_v\,v)/24 \\
e_{36} &= 36\,c_{16} + 6\,d_1 - 5\,n_v + (n_v\,v)/2 \\
e_{37} &= 72\,c_{16} + 18\,d_1 - 12\,e_1 - 6\,n_v + (n_v\,v)/2 \\
e_{38} &= 48\,c_{16} + 12\,d_1 + e_2 - (11\,n_v)/2 + (n_v\,v)/2 \\
e_{39} &= 36\,c_{16} + 6\,d_1 - 8\,e_1 + 2\,e_2 + 4\,e_3 - (7\,n_v)/3 + (n_v\,v)/6 \\
e_{40} &= 96\,c_{16} + 24\,d_1 - 11\,n_v + n_v\,v \\
e_{41} &= 144\,c_{16} + 36\,d_1 + 8\,e_2 + 12\,e_3 - 13\,n_v + n_v\,v \\
e_{42} &= (144\,c_{16} + 48\,d_1 + 4\,e_2 - 13\,n_v + n_v\,v)/2 \\
e_{43} &= (48\,c_{16} - 9\,n_v + n_v\,v)/8 \\
e_{44} &= 120\,c_{16} + 36\,d_1 + 2\,e_2 - 12\,n_v + n_v\,v \\
e_{45} &= (96\,c_{16} + 24\,e_3 - 11\,n_v + n_v\,v)/8 \\
e_{46} &= (144\,c_{16} + 48\,d_1 + 4\,e_2 - 13\,n_v + n_v\,v)/2 \\
e_{47} &= (120\,c_{16} + 24\,d_1 - 24\,e_1 - 11\,n_v + n_v\,v)/2 \\
e_{48} &= (48\,c_{16} - 9\,n_v + n_v\,v)/2 \\
e_{49} &= (192\,c_{16} + 48\,d_1 - 48\,e_1 + 4\,e_2 - 13\,n_v + n_v\,v)/2 \\
e_{50} &= 120\,c_{16} + 36\,d_1 + 2\,e_2 - 12\,n_v + n_v\,v \\
e_{51} &= (117\,c_{16} - 12\,e_1 - 22\,c_{16}\,v + c_{16}\,v^2)/2 \\
e_{52} &= 90\,c_{16} + 6\,d_1 - 24\,e_1 - (11\,n_v)/4 - 6\,c_{16}\,v + (n_v\,v)/4
\end{aligned}
$$

$$\begin{aligned}
e_{53} &= ((24\,c_{16} - n_v)\,(9 - v))/2 \\
e_{54} &= (312\,c_{16} + 24\,d_1 - 48\,e_1 - 11\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{55} &= 114\,c_{16} + 12\,d_1 - 36\,e_1 + e_2 - (13\,n_v)/4 - 6\,c_{16}\,v + (n_v\,v)/4 \\
e_{56} &= (360\,c_{16} + 48\,d_1 - 48\,e_1 + 4\,e_2 - 13\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{57} &= (96\,c_{16} + 24\,d_1 - 11\,n_v + n_v\,v)/2 \\
e_{58} &= (d_1\,(117 - 22\,v + v^2))/6 \\
e_{59} &= (312\,c_{16} + 132\,d_1 - 48\,e_1 + 4\,e_2 - 11\,n_v - 24\,c_{16}\,v - 12\,d_1\,v + n_v\,v)/2 \\
e_{60} &= ((-24\,c_{16} - 12\,d_1 + n_v)\,(-11 + v))/4 \\
e_{61} &= ((-24\,c_{16} - 12\,d_1 + n_v)\,(-9 + v))/4 \\
e_{62} &= (312\,c_{16} + 132\,d_1 - 48\,e_1 - 11\,n_v - 24\,c_{16}\,v - 12\,d_1\,v + n_v\,v)/2 \\
e_{63} &= 102\,c_{16} + 39\,d_1 - 24\,e_1 + 2\,e_2 - (13\,n_v)/4 - 6\,c_{16}\,v - 3\,d_1\,v + (n_v\,v)/4 \\
e_{64} &= 78\,c_{16} + 39\,d_1 + 2\,e_2 - (13\,n_v)/4 - 6\,c_{16}\,v - 3\,d_1\,v + (n_v\,v)/4 \\
e_{65} &= 96\,c_{16} + 30\,d_1 - 12\,e_1 + 4\,e_2 - 7\,n_v + (n_v\,v)/2 \\
e_{66} &= (3\,c_{16}\,(63 - 16\,v + v^2))/4 \\
e_{67} &= 3\,(81\,c_{16} - 4\,e_1 - 18\,c_{16}\,v + c_{16}\,v^2) \\
e_{68} &= (3\,c_{16}\,(77 - 18\,v + v^2))/4 \\
e_{69} &= 3\,(95\,c_{16} - 4\,e_1 - 20\,c_{16}\,v + c_{16}\,v^2) \\
e_{70} &= 312\,c_{16} + 24\,d_1 - 48\,e_1 + 2\,e_2 - 11\,n_v - 24\,c_{16}\,v + n_v\,v \\
e_{71} &= 144\,c_{16} + 18\,d_1 + 2\,e_2 - 6\,n_v - 12\,c_{16}\,v + (n_v\,v)/2 \\
e_{72} &= 144\,c_{16} + 12\,d_1 - 12\,e_1 + e_2 - (11\,n_v)/2 - 12\,c_{16}\,v + (n_v\,v)/2 \\
e_{73} &= (288\,c_{16} + 24\,d_1 - 24\,e_1 - 11\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{74} &= 120\,c_{16} + 6\,e_3 - 5\,n_v - 12\,c_{16}\,v + (n_v\,v)/2 \\
e_{75} &= 336\,c_{16} + 36\,d_1 - 24\,e_1 + 6\,e_2 + 12\,e_3 - 13\,n_v - 24\,c_{16}\,v + n_v\,v \\
e_{76} &= 312\,c_{16} + 36\,d_1 - 24\,e_1 + 2\,e_2 - 12\,n_v - 24\,c_{16}\,v + n_v\,v \\
e_{77} &= (264\,c_{16} + 24\,d_1 + 4\,e_2 - 11\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{78} &= ((24\,c_{16} - n_v)\,(9 - v))/4 \\
e_{79} &= (264\,c_{16} + 24\,e_3 - 11\,n_v - 24\,c_{16}\,v + n_v\,v)/4 \\
e_{80} &= (264\,c_{16} + 24\,d_1 - 11\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{81} &= (312\,c_{16} + 24\,d_1 + 8\,e_2 + 24\,e_3 - 13\,n_v - 24\,c_{16}\,v + n_v\,v)/2 \\
e_{82} &= ((-24\,c_{16} - 12\,d_1 + n_v)\,(-11 + v))/8 \\
e_{83} &= (264\,c_{16} + 132\,d_1 + 4\,e_2 - 11\,n_v - 24\,c_{16}\,v - 12\,d_1\,v + n_v\,v)/2 \\
e_{84} &= ((-24\,c_{16} - 12\,d_1 + n_v)\,(-11 + v))/4 \\
e_{85} &= 39\,c_{16} + (39\,d_1)/2 + e_2 - (13\,n_v)/8 - 3\,c_{16}\,v - (3\,d_1\,v)/2 + (n_v\,v)/8 \\
e_{86} &= 84\,c_{16} + 9\,d_1 + 3\,e_2 + 6\,e_3 - (7\,n_v)/2 - 6\,c_{16}\,v + (n_v\,v)/4
\end{aligned}$$

$$
\begin{aligned}
e_{87} &= 24\,c_{16} + 6\,d_1 - 4\,e_1 - 2\,n_v + (n_v\,v)/6 \\
e_{88} &= (144\,c_{16} + 48\,d_1 + 6\,e_2 - 12\,e_4 - 13\,n_v + n_v\,v)/2 \\
e_{89} &= 42\,c_{16} + 12\,d_1 + 2\,e_2 + 3\,e_3 - 3\,e_4 - (7\,n_v)/2 + (n_v\,v)/4 \\
e_{90} &= 15\,c_{16} + 6\,d_1 - 9\,e_5 - (3\,n_v)/2 + (n_v\,v)/8 \\
e_{91} &= -270\,c_{16} - 84\,d_1 - 2\,e_2 + 34\,n_v + 18\,c_{16}\,v + 6\,d_1\,v - (23\,n_v\,v)/4 + (n_v\,v^2)/4 \\
e_{92} &= -66\,c_{16} - 2\,e_3 + (55\,n_v)/6 + 6\,c_{16}\,v - (7\,n_v\,v)/4 + (n_v\,v^2)/12 \\
e_{93} &= (-984\,c_{16} - 132\,d_1 + 48\,e_1 - 4\,e_2 + 137\,n_v + 48\,c_{16}\,v - 23\,n_v\,v + n_v\,v^2)/2 \\
e_{94} &= -42\,c_{16} - 6\,d_1 + 14\,n_v - (21\,n_v\,v)/8 + (n_v\,v^2)/8 \\
e_{95} &= (-312\,c_{16} + 90\,n_v + 24\,c_{16}\,v - 19\,n_v\,v + n_v\,v^2)/4 \\
e_{96} &= -213\,c_{16} - 18\,d_1 + 30\,e_1 - e_2 + (69\,n_v)/4 + 15\,c_{16}\,v - (23\,n_v\,v)/8 + (n_v\,v^2)/8 \\
e_{97} &= -138\,c_{16} - 12\,d_1 + 12\,e_1 + 28\,n_v + 6\,c_{16}\,v - (21\,n_v\,v)/4 + (n_v\,v^2)/4 \\
e_{98} &= -588\,c_{16} - 60\,d_1 + 48\,e_1 - 2\,e_2 + 68\,n_v + 36\,c_{16}\,v - (23\,n_v\,v)/2 + (n_v\,v^2)/2 \\
e_{99} &= -324\,c_{16} - 42\,d_1 + 12\,e_1 - e_2 + 62\,n_v + 12\,c_{16}\,v - 11\,n_v\,v + (n_v\,v^2)/2 \\
e_{100} &= -210\,c_{16} - 39\,d_1 + 28\,n_v + 18\,c_{16}\,v + 3\,d_1\,v - (21\,n_v\,v)/4 + (n_v\,v^2)/4 \\
e_{101} &= -780\,c_{16} - 114\,d_1 + 72\,e_1 - 4\,e_2 + 68\,n_v + 60\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (23\,n_v\,v)/2 + (n_v\,v^2)/2 \\
e_{102} &= (-1080\,c_{16} - 204\,d_1 + 48\,e_1 + 134\,n_v + 72\,c_{16}\,v + 12\,d_1\,v - 23\,n_v\,v + n_v\,v^2)/2 \\
e_{103} &= -996\,c_{16} - 162\,d_1 + 96\,e_1 - 16\,e_2 - 24\,e_3 + 83\,n_v + 60\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (25\,n_v\,v)/2 + (n_v\,v^2)/2 \\
e_{104} &= -876\,c_{16} - 150\,d_1 + 24\,e_1 - 10\,e_2 - 12\,e_3 + 81\,n_v + 60\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (25\,n_v\,v)/2 + (n_v\,v^2)/2 \\
e_{105} &= -372\,c_{16} - 96\,d_1 + 12\,e_1 - 6\,e_2 - 6\,e_3 + (151\,n_v)/4 + 24\,c_{16}\,v + 6\,d_1\,v \\
&\quad - 6\,n_v\,v + (n_v\,v^2)/4 \\
e_{106} &= (-1344\,c_{16} - 360\,d_1 - 8\,e_2 + 147\,n_v + 96\,c_{16}\,v + 24\,d_1\,v - 24\,n_v\,v + n_v\,v^2)/2 \\
e_{107} &= -1296\,c_{16} - 168\,d_1 + 48\,e_1 - 14\,e_2 - 12\,e_3 + 150\,n_v + 72\,c_{16}\,v - 24\,n_v\,v + n_v\,v^2 \\
e_{108} &= ((-11 + v)\,(48\,c_{16} - 9\,n_v + n_v\,v))/4 \\
e_{109} &= (-816\,c_{16} - 96\,d_1 - 4\,e_2 + 125\,n_v + 48\,c_{16}\,v - 22\,n_v\,v + n_v\,v^2)/2 \\
e_{110} &= (-1440\,c_{16} - 168\,d_1 - 16\,e_2 - 24\,e_3 + 151\,n_v + 96\,c_{16}\,v - 24\,n_v\,v + n_v\,v^2)/2 \\
e_{111} &= (-1056\,c_{16} - 48\,e_3 + 121\,n_v + 96\,c_{16}\,v - 22\,n_v\,v + n_v\,v^2)/4 \\
e_{112} &= (-768\,c_{16} - 72\,d_1 + 123\,n_v + 48\,c_{16}\,v - 22\,n_v\,v + n_v\,v^2)/4 \\
e_{113} &= -288\,c_{16} - 42\,d_1 - 6\,e_2 - 12\,e_3 + (153\,n_v)/4 + 12\,c_{16}\,v - 6\,n_v\,v + (n_v\,v^2)/4 \\
e_{114} &= -264\,c_{16} - 42\,d_1 - 2\,e_2 + (149\,n_v)/4 + 12\,c_{16}\,v - 6\,n_v\,v + (n_v\,v^2)/4 \\
e_{115} &= (-1104\,c_{16} - 168\,d_1 + 48\,e_1 - 8\,e_2 + 149\,n_v + 48\,c_{16}\,v - 24\,n_v\,v + n_v\,v^2)/2
\end{aligned}
$$

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\begin{aligned}
e_{116} &= -408\,c_{16} - 48\,d_1 + 24\,e_1 - 4\,e_2 - 6\,e_3 + (81\,n_v)/2 + 24\,c_{16}\,v \\
&\quad - (25\,n_v\,v)/4 + (n_v\,v^2)/4 \\
e_{117} &= ((-9+v)\,(96\,c_{16} - 11\,n_v + n_v\,v))/16 \\
e_{118} &= (-1344\,c_{16} - 96\,d_1 + 192\,e_1 + 125\,n_v + 96\,c_{16}\,v - 22\,n_v\,v + n_v\,v^2)/8 \\
e_{119} &= ((-11+v)\,(48\,c_{16} - 9\,n_v + n_v\,v))/8 \\
e_{120} &= -270\,c_{16} - 51\,d_1 + 48\,e_1 - 2\,e_2 + (151\,n_v)/8 + 18\,c_{16}\,v + 3\,d_1\,v - 3\,n_v\,v + (n_v\,v^2)/8 \\
e_{121} &= -444\,c_{16} - 90\,d_1 + 24\,e_1 - 2\,e_2 + (147\,n_v)/4 + 36\,c_{16}\,v + 6\,d_1\,v - 6\,n_v\,v \\
&\quad + (n_v\,v^2)/4 \\
e_{122} &= -288\,c_{16} - 42\,d_1 + 24\,e_1 - 2\,e_2 + (149\,n_v)/4 + 12\,c_{16}\,v - 6\,n_v\,v + (n_v\,v^2)/4 \\
e_{123} &= -360\,c_{16} - 18\,d_1 + 36\,e_1 - e_2 + (27\,n_v)/2 + 50\,c_{16}\,v - (25\,n_v\,v)/12 \\
&\quad - 2\,c_{16}\,v^2 + (n_v\,v^2)/12 \\
e_{124} &= (n_v\,(63 - 16\,v + v^2))/48 \\
e_{125} &= ((-11+v)\,(96\,c_{16} + 24\,d_1 - 11\,n_v + n_v\,v))/8 \\
e_{126} &= (-432\,c_{16} - 72\,d_1 + 123\,n_v - 22\,n_v\,v + n_v\,v^2)/12 \\
e_{127} &= -348\,c_{16} - 162\,d_1 + 24\,e_1 - 2\,e_2 + (27\,n_v)/2 + 50\,c_{16}\,v + 25\,d_1\,v \\
&\quad - (25\,n_v\,v)/12 - 2\,c_{16}\,v^2 - d_1\,v^2 + (n_v\,v^2)/12 \\
e_{128} &= -132\,c_{16} - 33\,d_1 + 12\,e_1 + (143\,n_v)/16 + 12\,c_{16}\,v + 3\,d_1\,v - (3\,n_v\,v)/2 + (n_v\,v^2)/16 \\
e_{129} &= -348\,c_{16} - 90\,d_1 + 48\,e_1 - 4\,e_2 + (177\,n_v)/8 + 24\,c_{16}\,v + 6\,d_1\,v - (13\,n_v\,v)/4 \\
&\quad + (n_v\,v^2)/8 \\
e_{130} &= -444\,c_{16} - 120\,d_1 + 24\,e_1 - 6\,e_2 + (175\,n_v)/4 + 24\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (13\,n_v\,v)/2 + (n_v\,v^2)/4 \\
e_{131} &= (-1158\,c_{16} + 24\,e_1 + 337\,c_{16}\,v - 32\,c_{16}\,v^2 + c_{16}\,v^3)/2 \\
e_{132} &= (-1326\,c_{16} + 24\,e_1 + 375\,c_{16}\,v - 34\,c_{16}\,v^2 + c_{16}\,v^3)/8 \\
e_{133} &= ((-9+v)\,(264\,c_{16} + 48\,d_1 - 11\,n_v - 24\,c_{16}\,v + n_v\,v))/32 \\
e_{134} &= -798\,c_{16} - 78\,d_1 + 48\,e_1 - 2\,e_2 + (125\,n_v)/4 + 132\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (11\,n_v\,v)/2 - 6\,c_{16}\,v^2 + (n_v\,v^2)/4 \\
e_{135} &= -441\,c_{16} - 72\,d_1 - 2\,e_2 + (147\,n_v)/8 + 72\,c_{16}\,v + 6\,d_1\,v - 3\,n_v\,v \\
&\quad - 3\,c_{16}\,v^2 + (n_v\,v^2)/8 \\
e_{136} &= -471\,c_{16} - 48\,d_1 + 24\,e_1 - 2\,e_2 + (149\,n_v)/8 + 72\,c_{16}\,v + 3\,d_1\,v - 3\,n_v\,v \\
&\quad - 3\,c_{16}\,v^2 + (n_v\,v^2)/8 \\
e_{137} &= ((-24\,c_{16} + n_v)\,(99 - 20\,v + v^2))/8 \\
e_{138} &= -387\,c_{16} - 12\,d_1 + 12\,e_1 - e_2 + (125\,n_v)/8 + 66\,c_{16}\,v - (11\,n_v\,v)/4 \\
&\quad - 3\,c_{16}\,v^2 + (n_v\,v^2)/8
\end{aligned}
$$

$$\begin{aligned}
e_{139} &= -387\,c_{16} - 12\,d_1 + 12\,e_1 + (125\,n_v)/8 + 66\,c_{16}\,v - (11\,n_v\,v)/4 - 3\,c_{16}\,v^2 \\
&\quad + (n_v\,v^2)/8 \\
e_{140} &= (-2904\,c_{16} - 48\,e_3 + 121\,n_v + 528\,c_{16}\,v - 22\,n_v\,v - 24\,c_{16}\,v^2 + n_v\,v^2)/8 \\
e_{141} &= -930\,c_{16} - 90\,d_1 + 24\,e_1 - 6\,e_2 - 12\,e_3 + (151\,n_v)/4 + 144\,c_{16}\,v \\
&\quad + 6\,d_1\,v - 6\,n_v\,v - 6\,c_{16}\,v^2 + (n_v\,v^2)/4 \\
e_{142} &= -906\,c_{16} - 90\,d_1 + 24\,e_1 - 2\,e_2 + (147\,n_v)/4 + 144\,c_{16}\,v + 6\,d_1\,v \\
&\quad - 6\,n_v\,v - 6\,c_{16}\,v^2 + (n_v\,v^2)/4 \\
e_{143} &= -1860\,c_{16} - 72\,d_1 + 72\,e_1 - 6\,e_2 - 12\,e_3 + (149\,n_v)/2 + 288\,c_{16}\,v \\
&\quad - 12\,n_v\,v - 12\,c_{16}\,v^2 + (n_v\,v^2)/2 \\
e_{144} &= -324\,c_{16} - 12\,d_1 - 2\,e_2 - 6\,e_3 + (27\,n_v)/2 + 50\,c_{16}\,v - (25\,n_v\,v)/12 \\
&\quad - 2\,c_{16}\,v^2 + (n_v\,v^2)/12 \\
e_{145} &= -162\,c_{16} - 81\,d_1 - e_2 + (27\,n_v)/4 + 25\,c_{16}\,v + (25\,d_1\,v)/2 - (25\,n_v\,v)/24 \\
&\quad - c_{16}\,v^2 - (d_1\,v^2)/2 + (n_v\,v^2)/24 \\
e_{146} &= ((-11 + v)\,(312\,c_{16} + 48\,d_1 - 13\,n_v - 24\,c_{16}\,v + n_v\,v))/16 \\
e_{147} &= (-531\,c_{16})/2 - 39\,d_1 - 3\,e_2 - 6\,e_3 + (177\,n_v)/16 + 39\,c_{16}\,v + 3\,d_1\,v \\
&\quad - (13\,n_v\,v)/8 - (3\,c_{16}\,v^2)/2 + (n_v\,v^2)/16 \\
e_{148} &= -1074\,c_{16} - 108\,d_1 + 24\,e_1 - 8\,e_2 - 12\,e_3 + (175\,n_v)/4 + 156\,c_{16}\,v \\
&\quad + 6\,d_1\,v - (13\,n_v\,v)/2 - 6\,c_{16}\,v^2 + (n_v\,v^2)/4 \\
e_{149} &= (-1728\,c_{16} - 360\,d_1 + 48\,e_1 - 8\,e_2 + 147\,n_v + 144\,c_{16}\,v + 24\,d_1\,v \\
&\quad - 24\,n_v\,v + n_v\,v^2)/2 \\
e_{150} &= -528\,c_{16} - 120\,d_1 + 24\,e_1 - 6\,e_2 + 6\,e_4 + (175\,n_v)/4 + 36\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (13\,n_v\,v)/2 + (n_v\,v^2)/4 \\
e_{151} &= -336\,c_{16} - 84\,d_1 + 12\,e_1 + (145\,n_v)/4 + 24\,c_{16}\,v + 6\,d_1\,v - 6\,n_v\,v + (n_v\,v^2)/4 \\
e_{152} &= (-1728\,c_{16} - 480\,d_1 - 32\,e_2 - 24\,e_3 + 24\,e_4 + 177\,n_v + 96\,c_{16}\,v \\
&\quad + 24\,d_1\,v - 26\,n_v\,v + n_v\,v^2)/2 \\
e_{153} &= -900\,c_{16} - 126\,d_1 + 24\,e_1 - 17\,e_2 - 24\,e_3 + 6\,e_4 + 89\,n_v + 48\,c_{16}\,v \\
&\quad - 13\,n_v\,v + (n_v\,v^2)/2 \\
e_{154} &= -240\,c_{16} - 36\,d_1 + 6\,e_1 - 4\,e_2 - 6\,e_3 + 3\,e_4 + (95\,n_v)/4 + 12\,c_{16}\,v \\
&\quad - (27\,n_v\,v)/8 + (n_v\,v^2)/8 \\
e_{155} &= -306\,c_{16} - 60\,d_1 - 3\,e_2 + 3\,e_4 + 18\,e_5 + (163\,n_v)/4 + 12\,c_{16}\,v - (25\,n_v\,v)/4 \\
&\quad + (n_v\,v^2)/4 \\
e_{156} &= -282\,c_{16} - 60\,d_1 - 7\,e_2 - 6\,e_3 + 6\,e_4 + (83\,n_v)/2 + 6\,c_{16}\,v - (25\,n_v\,v)/4 \\
&\quad + (n_v\,v^2)/4
\end{aligned}$$

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\begin{aligned}
e_{157} &= -171\,c_{16} - 57\,d_1 - e_2 + 18\,e_5 + (81\,n_v)/4 + 9\,c_{16}\,v + 3\,d_1\,v - (25\,n_v\,v)/8 \\
&\quad + (n_v\,v^2)/8 \\
e_{158} &= ((-11+v)\,(-408\,c_{16} + 108\,n_v + 24\,c_{16}\,v - 21\,n_v\,v + n_v\,v^2))/16 \\
e_{159} &= (10920\,c_{16} + 1968\,d_1 + 32\,e_2 - 1820\,n_v - 1152\,c_{16}\,v - 144\,d_1\,v \\
&\quad + 439\,n_v\,v + 24\,c_{16}\,v^2 - 36\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{160} &= 819\,c_{16} + 48\,d_1 - 12\,e_1 + e_2 - (377\,n_v)/2 - 90\,c_{16}\,v + (389\,n_v\,v)/8 \\
&\quad + 3\,c_{16}\,v^2 - (17\,n_v\,v^2)/4 + (n_v\,v^3)/8 \\
e_{161} &= 2970\,c_{16} + 378\,d_1 - 48\,e_1 + 18\,e_2 + 24\,e_3 - 470\,n_v - 288\,c_{16}\,v - 18\,d_1\,v \\
&\quad + (443\,n_v\,v)/4 + 6\,c_{16}\,v^2 - 9\,n_v\,v^2 + (n_v\,v^3)/4 \\
e_{162} &= 2442\,c_{16} + 192\,d_1 - 60\,e_1 + 10\,e_2 + 12\,e_3 - 458\,n_v - 228\,c_{16}\,v \\
&\quad + (439\,n_v\,v)/4 + 6\,c_{16}\,v^2 - 9\,n_v\,v^2 + (n_v\,v^3)/4 \\
e_{163} &= (7128\,c_{16} + 144\,e_3 - 1452\,n_v - 912\,c_{16}\,v + 385\,n_v\,v + 24\,c_{16}\,v^2 \\
&\quad - 34\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{164} &= 2634\,c_{16} + 342\,d_1 - 48\,e_1 + 6\,e_2 - 456\,n_v - 264\,c_{16}\,v - 18\,d_1\,v + (439\,n_v\,v)/4 \\
&\quad + 6\,c_{16}\,v^2 - 9\,n_v\,v^2 + (n_v\,v^3)/4 \\
e_{165} &= 2022\,c_{16} + 309\,d_1 - 48\,e_1 + 6\,e_2 - (1813\,n_v)/4 - 138\,c_{16}\,v - 15\,d_1\,v \\
&\quad + (219\,n_v\,v)/2 - 9\,n_v\,v^2 + (n_v\,v^3)/4 \\
e_{166} &= (n_v\,(-693 + 239\,v - 27\,v^2 + v^3))/96 \\
e_{167} &= ((-11+v)\,(-192\,c_{16} + 99\,n_v - 20\,n_v\,v + n_v\,v^2))/16 \\
e_{168} &= ((-11+v)\,(-816\,c_{16} - 96\,d_1 + 125\,n_v + 48\,c_{16}\,v - 22\,n_v\,v + n_v\,v^2))/16 \\
e_{169} &= ((48\,c_{16} - 9\,n_v + n_v\,v)\,(143 - 24\,v + v^2))/32 \\
e_{170} &= (3504\,c_{16} + 408\,d_1 - 1357\,n_v - 240\,c_{16}\,v - 24\,d_1\,v + 365\,n_v\,v \\
&\quad - 33\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{171} &= 429\,c_{16} + 30\,d_1 - 30\,e_1 + e_2 - (301\,n_v)/4 - 40\,c_{16}\,v + (437\,n_v\,v)/24 + c_{16}\,v^2 \\
&\quad - (3\,n_v\,v^2)/2 + (n_v\,v^3)/24 \\
e_{172} &= (14928\,c_{16} + 1632\,d_1 - 1152\,e_1 + 48\,e_2 - 1729\,n_v - 1728\,c_{16}\,v - 96\,d_1\,v \\
&\quad + 419\,n_v\,v + 48\,c_{16}\,v^2 - 35\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{173} &= (4512\,c_{16} + 288\,d_1 - 384\,e_1 - 1383\,n_v - 288\,c_{16}\,v + 367\,n_v\,v \\
&\quad - 33\,n_v\,v^2 + n_v\,v^3)/48 \\
e_{174} &= (11568\,c_{16} + 1008\,d_1 - 384\,e_1 + 16\,e_2 - 1677\,n_v - 1440\,c_{16}\,v - 48\,d_1\,v \\
&\quad + 415\,n_v\,v + 48\,c_{16}\,v^2 - 35\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{175} &= (6576\,c_{16} + 744\,d_1 - 288\,e_1 + 16\,e_2 - 1655\,n_v - 432\,c_{16}\,v - 24\,d_1\,v \\
&\quad + 413\,n_v\,v - 35\,n_v\,v^2 + n_v\,v^3)/4 \\
e_{176} &= 1674\,c_{16} + 279\,d_1 - 72\,e_1 + 10\,e_2 + 12\,e_3 - 184\,n_v - 192\,c_{16}\,v - 28\,d_1\,v \\
&\quad + (493\,n_v\,v)/12 + 6\,c_{16}\,v^2 + d_1\,v^2 - (19\,n_v\,v^2)/6 + (n_v\,v^3)/12
\end{aligned}
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\begin{aligned}
e_{177} &= 1062\,c_{16} + 90\,d_1 - 48\,e_1 + 2\,e_2 - (1677\,n_v)/16 - 156\,c_{16}\,v - 6\,d_1\,v \\
&\quad + (415\,n_v\,v)/16 + 6\,c_{16}\,v^2 - (35\,n_v\,v^2)/16 + (n_v\,v^3)/16 \\
e_{178} &= 1392\,c_{16} + 186\,d_1 - 72\,e_1 + 10\,e_2 + 12\,e_3 - (2091\,n_v)/16 - 174\,c_{16}\,v \\
&\quad - 12\,d_1\,v + (471\,n_v\,v)/16 + 6\,c_{16}\,v^2 - (37\,n_v\,v^2)/16 + (n_v\,v^3)/16 \\
e_{179} &= (19680\,c_{16} + 2256\,d_1 - 192\,e_1 + 80\,e_2 + 96\,e_3 - 2023\,n_v - 2688\,c_{16}\,v \\
&\quad - 144\,d_1\,v + 467\,n_v\,v + 96\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{180} &= 4176\,c_{16} + 450\,d_1 - 192\,e_1 + 24\,e_2 + 24\,e_3 - (2057\,n_v)/4 - 444\,c_{16}\,v \\
&\quad - 18\,d_1\,v + (469\,n_v\,v)/4 + 12\,c_{16}\,v^2 - (37\,n_v\,v^2)/4 + (n_v\,v^3)/4 \\
e_{181} &= (14880\,c_{16} + 2040\,d_1 - 288\,e_1 + 72\,e_2 + 48\,e_3 - 2001\,n_v - 1680\,c_{16}\,v \\
&\quad - 120\,d_1\,v + 465\,n_v\,v + 48\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/4 \\
e_{182} &= 1116\,c_{16} + 78\,d_1 + 6\,e_2 + 12\,e_3 - (669\,n_v)/4 - 124\,c_{16}\,v + (155\,n_v\,v)/4 \\
&\quad + 4\,c_{16}\,v^2 - (37\,n_v\,v^2)/12 + (n_v\,v^3)/12 \\
e_{183} &= (13728\,c_{16} + 192\,e_3 - 1573\,n_v - 2304\,c_{16}\,v + 407\,n_v\,v + 96\,c_{16}\,v^2 \\
&\quad - 35\,n_v\,v^2 + n_v\,v^3)/32 \\
e_{184} &= (14400\,c_{16} + 1176\,d_1 - 384\,e_1 + 64\,e_2 + 96\,e_3 - 2005\,n_v - 1584\,c_{16}\,v - 24\,d_1\,v \\
&\quad + 465\,n_v\,v + 48\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{185} &= (12864\,c_{16} + 1464\,d_1 - 192\,e_1 + 32\,e_2 - 1949\,n_v - 1488\,c_{16}\,v - 72\,d_1\,v \\
&\quad + 461\,n_v\,v + 48\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{186} &= (6432\,c_{16} + 816\,d_1 - 192\,e_1 - 1625\,n_v - 480\,c_{16}\,v - 48\,d_1\,v + 411\,n_v\,v \\
&\quad - 35\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{187} &= 822\,c_{16} + 105\,d_1 - 48\,e_1 + 8\,e_2 + 12\,e_3 - (2091\,n_v)/16 - 54\,c_{16}\,v - 3\,d_1\,v \\
&\quad + (471\,n_v\,v)/16 - (37\,n_v\,v^2)/16 + (n_v\,v^3)/16 \\
e_{188} &= (9360\,c_{16} + 1152\,d_1 - 288\,e_1 + 56\,e_2 + 48\,e_3 - 1981\,n_v - 576\,c_{16}\,v \\
&\quad - 24\,d_1\,v + 463\,n_v\,v - 37\,n_v\,v^2 + n_v\,v^3)/4 \\
e_{189} &= 684\,c_{16} + 69\,d_1 - 48\,e_1 + 2\,e_2 - (667\,n_v)/8 - 74\,c_{16}\,v - 3\,d_1\,v \\
&\quad + (155\,n_v\,v)/8 + 2\,c_{16}\,v^2 - (37\,n_v\,v^2)/24 + (n_v\,v^3)/24 \\
e_{190} &= ((-13 + v)\,(-432\,c_{16} - 72\,d_1 + 123\,n_v - 22\,n_v\,v + n_v\,v^2))/72 \\
e_{191} &= 1110\,c_{16} + 270\,d_1 - 72\,e_1 + 6\,e_2 - (801\,n_v)/8 - 130\,c_{16}\,v - 31\,d_1\,v \\
&\quad + (521\,n_v\,v)/24 + 4\,c_{16}\,v^2 + d_1\,v^2 - (13\,n_v\,v^2)/8 + (n_v\,v^3)/24 \\
e_{192} &= 282\,c_{16} + 57\,d_1 - 12\,e_1 + 2\,e_2 - (775\,n_v)/16 - 18\,c_{16}\,v - 3\,d_1\,v \\
&\quad + (515\,n_v\,v)/48 - (13\,n_v\,v^2)/16 + (n_v\,v^3)/48
\end{aligned}
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\begin{aligned}
e_{193} &= (18486\,c_{16} - 216\,e_1 - 6453\,c_{16}\,v + 829\,c_{16}\,v^2 - 47\,c_{16}\,v^3 + c_{16}\,v^4)/72 \\
e_{194} &= (49176\,c_{16} + 5760\,d_1 - 1152\,e_1 + 96\,e_2 - 2001\,n_v - 11160\,c_{16}\,v \\
&\quad - 744\,d_1\,v + 465\,n_v\,v + 888\,c_{16}\,v^2 + 24\,d_1\,v^2 - 37\,n_v\,v^2 - 24\,c_{16}\,v^3 \\
&\quad + n_v\,v^3)/48 \\
e_{195} &= 4074\,c_{16} + 126\,d_1 - 72\,e_1 + 6\,e_2 + 12\,e_3 - (667\,n_v)/4 - 930\,c_{16}\,v - 6\,d_1\,v \\
&\quad + (155\,n_v\,v)/4 + 74\,c_{16}\,v^2 - (37\,n_v\,v^2)/12 - 2\,c_{16}\,v^3 + (n_v\,v^3)/12 \\
e_{196} &= 2427\,c_{16} + 258\,d_1 - 24\,e_1 + 8\,e_2 + 12\,e_3 - (801\,n_v)/8 - 521\,c_{16}\,v \\
&\quad - 31\,d_1\,v + (521\,n_v\,v)/24 + 39\,c_{16}\,v^2 + d_1\,v^2 - (13\,n_v\,v^2)/8 - c_{16}\,v^3 \\
&\quad + (n_v\,v^3)/24 \\
e_{197} &= (56760\,c_{16} + 2544\,d_1 - 960\,e_1 + 128\,e_2 + 192\,e_3 - 2325\,n_v - 12360\,c_{16}\,v \\
&\quad - 144\,d_1\,v + 515\,n_v\,v + 936\,c_{16}\,v^2 - 39\,n_v\,v^2 - 24\,c_{16}\,v^3 + n_v\,v^3)/32 \\
e_{198} &= ((144\,c_{16} - 13\,n_v + n_v\,v)\,(99 - 20\,v + v^2))/96 \\
e_{199} &= (23472\,c_{16} + 2496\,d_1 - 384\,e_1 + 32\,e_2 - 1963\,n_v - 3648\,c_{16}\,v - 192\,d_1\,v \\
&\quad + 463\,n_v\,v + 144\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{200} &= (19536\,c_{16} + 1560\,d_1 - 576\,e_1 + 64\,e_2 + 96\,e_3 - 2001\,n_v - 2640\,c_{16}\,v \\
&\quad - 72\,d_1\,v + 465\,n_v\,v + 96\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/8 \\
e_{201} &= 1932\,c_{16} + 480\,d_1 - 24\,e_1 + 12\,e_2 + 6\,e_3 - 6\,e_4 - (801\,n_v)/4 - 236\,c_{16}\,v \\
&\quad - 56\,d_1\,v + (521\,n_v\,v)/12 + 8\,c_{16}\,v^2 + 2\,d_1\,v^2 - (13\,n_v\,v^2)/4 + (n_v\,v^3)/12 \\
e_{202} &= 625\,c_{16} + 90\,d_1 - 20\,e_1 + 3\,e_2 - 2\,e_4 - (807\,n_v)/16 - 84\,c_{16}\,v - 6\,d_1\,v \\
&\quad + (523\,n_v\,v)/48 + 3\,c_{16}\,v^2 - (13\,n_v\,v^2)/16 + (n_v\,v^3)/48 \\
e_{203} &= (24144\,c_{16} + 3096\,d_1 - 480\,e_1 + 168\,e_2 + 192\,e_3 - 48\,e_4 - 2407\,n_v \\
&\quad - 2928\,c_{16}\,v - 168\,d_1\,v + 521\,n_v\,v + 96\,c_{16}\,v^2 - 39\,n_v\,v^2 + n_v\,v^3)/4 \\
e_{204} &= (13296\,c_{16} + 1872\,d_1 - 192\,e_1 - 1911\,n_v - 1632\,c_{16}\,v - 144\,d_1\,v \\
&\quad + 459\,n_v\,v + 48\,c_{16}\,v^2 - 37\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{205} &= 1221\,c_{16} + 189\,d_1 - 12\,e_1 + 11\,e_2 + 12\,e_3 - 6\,e_4 - (2437\,n_v)/16 \\
&\quad - 120\,c_{16}\,v - 9\,d_1\,v + (523\,n_v\,v)/16 + 3\,c_{16}\,v^2 - (39\,n_v\,v^2)/16 + (n_v\,v^3)/16 \\
e_{206} &= (18528\,c_{16} + 2208\,d_1 - 384\,e_1 + 152\,e_2 + 192\,e_3 - 48\,e_4 - 2387\,n_v \\
&\quad - 1824\,c_{16}\,v - 72\,d_1\,v + 519\,n_v\,v + 48\,c_{16}\,v^2 - 39\,n_v\,v^2 + n_v\,v^3)/4 \\
e_{207} &= 208\,c_{16} + 24\,d_1 - 6\,e_1 + 2\,e_2 + 3\,e_3 - e_4 - (141\,n_v)/4 - 12\,c_{16}\,v \\
&\quad + (181\,n_v\,v)/24 - (5\,n_v\,v^2)/9 + (n_v\,v^3)/72 \\
e_{208} &= 3162\,c_{16} + 513\,d_1 - 60\,e_1 + 36\,e_2 + 36\,e_3 - 18\,e_4 - (2253\,n_v)/4 \\
&\quad - 186\,c_{16}\,v - 15\,d_1\,v + 124\,n_v\,v - (19\,n_v\,v^2)/2 + (n_v\,v^3)/4
\end{aligned}
$$

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\begin{aligned}
e_{209} &= 1863\,c_{16} + 354\,d_1 + 12\,e_2 + 6\,e_3 - 6\,e_4 - 36\,e_5 - (563\,n_v)/2 - 162\,c_{16}\,v \\
&\quad - 18\,d_1\,v + (497\,n_v\,v)/8 + 3\,c_{16}\,v^2 - (19\,n_v\,v^2)/4 + (n_v\,v^3)/8 \\
e_{210} &= (14616\,c_{16} + 2832\,d_1 + 176\,e_2 + 192\,e_3 - 96\,e_4 - 2268\,n_v - 1200\,c_{16}\,v \\
&\quad - 144\,d_1\,v + 497\,n_v\,v + 24\,c_{16}\,v^2 - 38\,n_v\,v^2 + n_v\,v^3)/16 \\
e_{211} &= 3414\,c_{16} + 510\,d_1 - 48\,e_1 + 22\,e_2 + 12\,e_3 - 12\,e_4 - 36\,e_5 - (1103\,n_v)/2 \\
&\quad - 288\,c_{16}\,v - 18\,d_1\,v + (493\,n_v\,v)/4 + 6\,c_{16}\,v^2 - (19\,n_v\,v^2)/2 + (n_v\,v^3)/4 \\
e_{212} &= 237\,c_{16} + 77\,d_1 + e_2 - 6\,e_5 - (367\,n_v)/12 - 28\,c_{16}\,v - (28\,d_1\,v)/3 \\
&\quad + (493\,n_v\,v)/72 + c_{16}\,v^2 + (d_1\,v^2)/3 - (19\,n_v\,v^2)/36 + (n_v\,v^3)/72 \\
e_{213} &= 906\,c_{16} + 114\,d_1 - 16\,e_1 + 9\,e_2 + 10\,e_3 - 3\,e_4 - 6\,e_5 - (367\,n_v)/2 \\
&\quad - 48\,c_{16}\,v + 41\,n_v\,v - (19\,n_v\,v^2)/6 + (n_v\,v^3)/12 \\
e_{214} &= (n_v\,(-11 + v)\,(-9 + v)\,(-7 + v)\,(-5 + v))/46080 \\
e_{215} &= (n_v\,(9009 - 3800\,v + 590\,v^2 - 40\,v^3 + v^4))/768 \\
e_{216} &= ((-11 + v)\,(1152\,c_{16} - 1287\,n_v + 359\,n_v\,v - 33\,n_v\,v^2 + n_v\,v^3))/384 \\
e_{217} &= ((-11 + v)\,(7488\,c_{16} + 576\,d_1 - 1677\,n_v - 576\,c_{16}\,v + 415\,n_v\,v \\
&\quad - 35\,n_v\,v^2 + n_v\,v^3))/384 \\
e_{218} &= (-34080\,c_{16} - 2688\,d_1 + 17891\,n_v + 3840\,c_{16}\,v + 192\,d_1\,v - 6146\,n_v\,v \\
&\quad - 96\,c_{16}\,v^2 + 796\,n_v\,v^2 - 46\,n_v\,v^3 + n_v\,v^4)/64 \\
e_{219} &= ((143 - 24\,v + v^2)\,(-288\,c_{16} + 117\,n_v - 22\,n_v\,v + n_v\,v^2))/96 \\
e_{220} &= (-1095\,c_{16})/2 - 45\,d_1 + 24\,e_1 - e_2 + (23217\,n_v)/256 + (159\,c_{16}\,v)/2 + 3\,d_1\,v \\
&\quad - (455\,n_v\,v)/16 - 3\,c_{16}\,v^2 + (439\,n_v\,v^2)/128 - (3\,n_v\,v^3)/16 + (n_v\,v^4)/256 \\
e_{221} &= -1950\,c_{16} - 144\,d_1 + 48\,e_1 - 2\,e_2 + (21849\,n_v)/32 + 216\,c_{16}\,v + 6\,d_1\,v \\
&\quad - 221\,n_v\,v - 6\,c_{16}\,v^2 + (435\,n_v\,v^2)/16 - (3\,n_v\,v^3)/2 + (n_v\,v^4)/32 \\
e_{222} &= -4692\,c_{16} - 462\,d_1 + 144\,e_1 - 20\,e_2 - 24\,e_3 + (28107\,n_v)/32 + 570\,c_{16}\,v \\
&\quad + 24\,d_1\,v - (4163\,n_v\,v)/16 - 18\,c_{16}\,v^2 + (239\,n_v\,v^2)/8 - (25\,n_v\,v^3)/16 \\
&\quad + (n_v\,v^4)/32 \\
e_{223} &= -4431\,c_{16} - 456\,d_1 + 48\,e_1 - 12\,e_2 - 12\,e_3 + (27159\,n_v)/32 + 600\,c_{16}\,v \\
&\quad + 30\,d_1\,v - (4103\,n_v\,v)/16 - 21\,c_{16}\,v^2 + (119\,n_v\,v^2)/4 - (25\,n_v\,v^3)/16 \\
&\quad + (n_v\,v^4)/32 \\
e_{224} &= -5142\,c_{16} - 336\,d_1 + 60\,e_1 - 12\,e_2 - 18\,e_3 + (2219\,n_v)/2 + 769\,c_{16}\,v + 12\,d_1\,v \\
&\quad - (677\,n_v\,v)/2 - 44\,c_{16}\,v^2 + (949\,n_v\,v^2)/24 + c_{16}\,v^3 - (25\,n_v\,v^3)/12 \\
&\quad + (n_v\,v^4)/24 \\
e_{225} &= -2202\,c_{16} - 192\,d_1 + 64\,e_1 - 8\,e_2 - 8\,e_3 + (27215\,n_v)/48 + 228\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (4103\,n_v\,v)/24 - 6\,c_{16}\,v^2 + (119\,n_v\,v^2)/6 - (25\,n_v\,v^3)/24 + (n_v\,v^4)/48
\end{aligned}
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\begin{aligned}
e_{226} &= -12156\,c_{16} - 1116\,d_1 + 216\,e_1 - 34\,e_2 - 36\,e_3 + (26499\,n_v)/8 \\
&\quad + 1320\,c_{16}\,v + 48\,d_1\,v - (4051\,n_v\,v)/4 - 36\,c_{16}\,v^2 + (237\,n_v\,v^2)/2 \\
&\quad - (25\,n_v\,v^3)/4 + (n_v\,v^4)/8 \\
e_{227} &= (-67584\,c_{16} - 768\,e_3 + 20449\,n_v + 10368\,c_{16}\,v - 6864\,n_v\,v - 384\,c_{16}\,v^2 \\
&\quad + 862\,n_v\,v^2 - 48\,n_v\,v^3 + n_v\,v^4)/128 \\
e_{228} &= -1911\,c_{16} - 183\,d_1 + 24\,e_1 - 2\,e_2 + (26347\,n_v)/64 + 258\,c_{16}\,v + 12\,d_1\,v \\
&\quad - (4047\,n_v\,v)/32 - 9\,c_{16}\,v^2 + (237\,n_v\,v^2)/16 - (25\,n_v\,v^3)/32 + (n_v\,v^4)/64 \\
e_{229} &= -5670\,c_{16} - 600\,d_1 + 72\,e_1 - 8\,e_2 + (25775\,n_v)/16 + 648\,c_{16}\,v + 36\,d_1\,v \\
&\quad - (3999\,n_v\,v)/8 - 18\,c_{16}\,v^2 + 59\,n_v\,v^2 - (25\,n_v\,v^3)/8 + (n_v\,v^4)/16 \\
e_{230} &= ((-13 + v)\,(3312\,c_{16} + 288\,d_1 - 1482\,n_v - 144\,c_{16}\,v + 387\,n_v\,v - 34\,n_v\,v^2 \\
&\quad + n_v\,v^3))/96 \\
e_{231} &= -1743\,c_{16} - 144\,d_1 + 72\,e_1 - 3\,e_2 + (4029\,n_v)/8 + 188\,c_{16}\,v + 6\,d_1\,v \\
&\quad - (2529\,n_v\,v)/16 - 5\,c_{16}\,v^2 + (303\,n_v\,v^2)/16 - (49\,n_v\,v^3)/48 + (n_v\,v^4)/48 \\
e_{232} &= -9426\,c_{16} - 972\,d_1 + 240\,e_1 - 28\,e_2 - 24\,e_3 + (5003\,n_v)/4 + 1574\,c_{16}\,v \\
&\quad + 86\,d_1\,v - (8785\,n_v\,v)/24 - 94\,c_{16}\,v^2 - 2\,d_1\,v^2 + (331\,n_v\,v^2)/8 + 2\,c_{16}\,v^3 \\
&\quad - (17\,n_v\,v^3)/8 + (n_v\,v^4)/24 \\
e_{233} &= -4077\,c_{16} - 246\,d_1 + 48\,e_1 - 8\,e_2 - 12\,e_3 + (4843\,n_v)/8 + 713\,c_{16}\,v + 12\,d_1\,v \\
&\quad - (2887\,n_v\,v)/16 - 45\,c_{16}\,v^2 + (989\,n_v\,v^2)/48 + c_{16}\,v^3 - (17\,n_v\,v^3)/16 \\
&\quad + (n_v\,v^4)/48 \\
e_{234} &= -5382\,c_{16} - 588\,d_1 + 168\,e_1 - 22\,e_2 - 24\,e_3 + (4929\,n_v)/4 + 574\,c_{16}\,v + 37\,d_1\,v \\
&\quad - (8723\,n_v\,v)/24 - 16\,c_{16}\,v^2 - d_1\,v^2 + (991\,n_v\,v^2)/24 - (17\,n_v\,v^3)/8 \\
&\quad + (n_v\,v^4)/24 \\
e_{235} &= -3279\,c_{16} - 354\,d_1 + 60\,e_1 - 10\,e_2 - 6\,e_3 + (1761\,n_v)/2 + 348\,c_{16}\,v + 18\,d_1\,v \\
&\quad - (8517\,n_v\,v)/32 - 9\,c_{16}\,v^2 + (983\,n_v\,v^2)/32 - (51\,n_v\,v^3)/32 + (n_v\,v^4)/32 \\
e_{236} &= -516\,c_{16} - 96\,d_1 + 24\,e_1 - 2\,e_2 + (327\,n_v)/4 + 62\,c_{16}\,v + (31\,d_1\,v)/3 \\
&\quad - (3305\,n_v\,v)/144 - 2\,c_{16}\,v^2 - (d_1\,v^2)/3 + (1075\,n_v\,v^2)/432 - (53\,n_v\,v^3)/432 \\
&\quad + (n_v\,v^4)/432 \\
e_{237} &= -2973\,c_{16} - 138\,d_1 + 30\,e_1 - 4\,e_2 - 6\,e_3 + (981\,n_v)/8 + (3305\,c_{16}\,v)/4 \\
&\quad + (31\,d_1\,v)/2 - (3305\,n_v\,v)/96 - (1075\,c_{16}\,v^2)/12 - (d_1\,v^2)/2 \\
&\quad + (1075\,n_v\,v^2)/288 + (53\,c_{16}\,v^3)/12 - (53\,n_v\,v^3)/288 - (c_{16}\,v^4)/12 \\
&\quad + (n_v\,v^4)/288 \\
e_{238} &= (-5877\,c_{16})/2 - 147\,d_1 + 48\,e_1 - 4\,e_2 - 6\,e_3 + (4839\,n_v)/16 + 591\,c_{16}\,v \\
&\quad + 9\,d_1\,v - (2887\,n_v\,v)/32 - (83\,c_{16}\,v^2)/2 + (989\,n_v\,v^2)/96 + c_{16}\,v^3 \\
&\quad - (17\,n_v\,v^3)/32 + (n_v\,v^4)/96
\end{aligned}
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\begin{aligned}
e_{239} &= -7539\,c_{16} - 828\,d_1 + 108\,e_1 - 25\,e_2 - 24\,e_3 + 6\,e_4 + (5977\,n_v)/8 + 1374\,c_{16}\,v \\
&\quad + 80\,d_1\,v - (9983\,n_v\,v)/48 - 89\,c_{16}\,v^2 - 2\,d_1\,v^2 + (359\,n_v\,v^2)/16 + 2\,c_{16}\,v^3 \\
&\quad - (53\,n_v\,v^3)/48 + (n_v\,v^4)/48 \\
e_{240} &= -11034\,c_{16} - 1272\,d_1 + 144\,e_1 - 44\,e_2 - 48\,e_3 + 12\,e_4 + (5903\,n_v)/4 \\
&\quad + 1748\,c_{16}\,v + 111\,d_1\,v - (3307\,n_v\,v)/8 - 100\,c_{16}\,v^2 - 3\,d_1\,v^2 \\
&\quad + (1075\,n_v\,v^2)/24 + 2\,c_{16}\,v^3 - (53\,n_v\,v^3)/24 + (n_v\,v^4)/24 \\
e_{241} &= -8187\,c_{16} - 678\,d_1 + 132\,e_1 - 30\,e_2 - 36\,e_3 + 6\,e_4 + (2171\,n_v)/2 \\
&\quad + (2625\,c_{16}\,v)/2 + 36\,d_1\,v - (9819\,n_v\,v)/32 - 75\,c_{16}\,v^2 + (1071\,n_v\,v^2)/32 \\
&\quad + (3\,c_{16}\,v^3)/2 - (53\,n_v\,v^3)/32 + (n_v\,v^4)/32 \\
e_{242} &= (-106704\,c_{16} - 11232\,d_1 + 1152\,e_1 + 26910\,n_v + 13824\,c_{16}\,v + 864\,d_1\,v \\
&\quad - 8349\,n_v\,v - 432\,c_{16}\,v^2 + 977\,n_v\,v^2 - 51\,n_v\,v^3 + n_v\,v^4)/288 \\
e_{243} &= -3357\,c_{16} - 348\,d_1 + 60\,e_1 - 19\,e_2 - 24\,e_3 + 6\,e_4 + (5747\,n_v)/8 \\
&\quad + 342\,c_{16}\,v + 12\,d_1\,v - (3257\,n_v\,v)/16 - 9\,c_{16}\,v^2 + (1069\,n_v\,v^2)/48 \\
&\quad - (53\,n_v\,v^3)/48 + (n_v\,v^4)/48 \\
e_{244} &= -1524\,c_{16} - 204\,d_1 + 6\,e_1 - 7\,e_2 - 6\,e_3 + 3\,e_4 + 9\,e_5 + (33625\,n_v)/128 \\
&\quad + 186\,c_{16}\,v + 12\,d_1\,v - (2369\,n_v\,v)/32 - 6\,c_{16}\,v^2 + (519\,n_v\,v^2)/64 \\
&\quad - (13\,n_v\,v^3)/32 + (n_v\,v^4)/128 \\
e_{245} &= -10416\,c_{16} - 1278\,d_1 + 132\,e_1 - 47\,e_2 - 36\,e_3 + 18\,e_4 + 36\,e_5 \\
&\quad + (32749\,n_v)/16 + 1188\,c_{16}\,v + 66\,d_1\,v - (2339\,n_v\,v)/4 - 36\,c_{16}\,v^2 \\
&\quad + (517\,n_v\,v^2)/8 - (13\,n_v\,v^3)/4 + (n_v\,v^4)/16 \\
e_{246} &= -10254\,c_{16} - 1260\,d_1 + 144\,e_1 - 64\,e_2 - 72\,e_3 + 24\,e_4 + (32997\,n_v)/16 \\
&\quad + 1092\,c_{16}\,v + 60\,d_1\,v - (2343\,n_v\,v)/4 - 30\,c_{16}\,v^2 + (517\,n_v\,v^2)/8 \\
&\quad - (13\,n_v\,v^3)/4 + (n_v\,v^4)/16 \\
e_{247} &= -7842\,c_{16} - 1176\,d_1 + 48\,e_1 - 32\,e_2 - 24\,e_3 + 12\,e_4 + 36\,e_5 + 1373\,n_v \\
&\quad + 1079\,c_{16}\,v + 105\,d_1\,v - 391\,n_v\,v - 54\,c_{16}\,v^2 - 3\,d_1\,v^2 + (345\,n_v\,v^2)/8 \\
&\quad + c_{16}\,v^3 - (13\,n_v\,v^3)/6 + (n_v\,v^4)/24 \\
e_{248} &= -2355\,c_{16} - 225\,d_1 + 27\,e_1 - 8\,e_2 - 6\,e_3 + 3\,e_4 + 9\,e_5 + (3933\,n_v)/8 \\
&\quad + (2505\,c_{16}\,v)/8 + 9\,d_1\,v - (573\,n_v\,v)/4 - (33\,c_{16}\,v^2)/2 + (1027\,n_v\,v^2)/64 \\
&\quad + (3\,c_{16}\,v^3)/8 - (13\,n_v\,v^3)/16 + (n_v\,v^4)/64 \\
e_{249} &= -2388\,c_{16} - 384\,d_1 + 36\,e_1 - 14\,e_2 - 12\,e_3 + 6\,e_4 + 546\,n_v + 234\,c_{16}\,v \\
&\quad + 31\,d_1\,v - (9353\,n_v\,v)/60 - 6\,c_{16}\,v^2 - d_1\,v^2 + (517\,n_v\,v^2)/30 \\
&\quad - (13\,n_v\,v^3)/15 + (n_v\,v^4)/60 \\
e_{250} &= -16476\,c_{16} - 1920\,d_1 + 240\,e_1 - 90\,e_2 - 90\,e_3 + 30\,e_4 + 36\,e_5 \\
&\quad + (32089\,n_v)/8 + 1560\,c_{16}\,v + 72\,d_1\,v - (2313\,n_v\,v)/2 - 36\,c_{16}\,v^2 \\
&\quad + (515\,n_v\,v^2)/4 - (13\,n_v\,v^3)/2 + (n_v\,v^4)/8
\end{aligned}
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\begin{aligned}
e_{251} &= (n_v\,(-135135 + 66009\,v - 12650\,v^2 + 1190\,v^3 - 55\,v^4 + v^5))/23040 \\
e_{252} &= ((-13 + v)\,(-29376\,c_{16} - 1728\,d_1 + 21069\,n_v + 1728\,c_{16}\,v - 6960\,n_v\,v \\
&\quad + 866\,n_v\,v^2 - 48\,n_v\,v^3 + n_v\,v^4))/576 \\
e_{253} &= ((143 - 24\,v + v^2)\,(2304\,c_{16} - 1755\,n_v + 447\,n_v\,v - 37\,n_v\,v^2 + n_v\,v^3))/3072 \\
e_{254} &= (1863\,c_{16})/2 + 63\,d_1 - 24\,e_1 + e_2 - (114645\,n_v)/256 - 113\,c_{16}\,v - 3\,d_1\,v \\
&\quad + (43883\,n_v\,v)/256 + (7\,c_{16}\,v^2)/2 - (3403\,n_v\,v^2)/128 + (799\,n_v\,v^3)/384 \\
&\quad - (21\,n_v\,v^4)/256 + (n_v\,v^5)/768 \\
e_{255} &= (10875\,c_{16})/2 + (897\,d_1)/2 - 96\,e_1 + 12\,e_2 + 12\,e_3 - (147103\,n_v)/128 \\
&\quad - 917\,c_{16}\,v - 35\,d_1\,v + (157165\,n_v\,v)/384 + (105\,c_{16}\,v^2)/2 + (d_1\,v^2)/2 \\
&\quad - (11489\,n_v\,v^2)/192 - c_{16}\,v^3 + (857\,n_v\,v^3)/192 - (65\,n_v\,v^4)/384 + (n_v\,v^5)/384 \\
e_{256} &= 13545\,c_{16} + 1128\,d_1 - 264\,e_1 + 34\,e_2 + 36\,e_3 - (142595\,n_v)/32 \\
&\quad - 1801\,c_{16}\,v - 67\,d_1\,v + (154457\,n_v\,v)/96 + 77\,c_{16}\,v^2 + d_1\,v^2 \\
&\quad - (11399\,n_v\,v^2)/48 - c_{16}\,v^3 + (285\,n_v\,v^3)/16 - (65\,n_v\,v^4)/96 + (n_v\,v^5)/96 \\
e_{257} &= 13380\,c_{16} + 894\,d_1 - 144\,e_1 + 20\,e_2 + 24\,e_3 - (137943\,n_v)/32 \\
&\quad - 2050\,c_{16}\,v - 48\,d_1\,v + (50583\,n_v\,v)/32 + 108\,c_{16}\,v^2 - (11309\,n_v\,v^2)/48 \\
&\quad - 2\,c_{16}\,v^3 + (853\,n_v\,v^3)/48 - (65\,n_v\,v^4)/96 + (n_v\,v^5)/96 \\
e_{258} &= (1002240\,c_{16} + 83712\,d_1 - 12288\,e_1 + 1792\,e_2 + 1536\,e_3 - 406137\,n_v \\
&\quad - 122880\,c_{16}\,v - 4608\,d_1\,v + 149913\,n_v\,v + 3840\,c_{16}\,v^2 - 22470\,n_v\,v^2 \\
&\quad + 1702\,n_v\,v^3 - 65\,n_v\,v^4 + n_v\,v^5)/128 \\
e_{259} &= (570240\,c_{16} + 4608\,e_3 - 306735\,n_v - 89856\,c_{16}\,v + 123409\,n_v\,v \\
&\quad + 3456\,c_{16}\,v^2 - 19794\,n_v\,v^2 + 1582\,n_v\,v^3 - 63\,n_v\,v^4 + n_v\,v^5)/4608 \\
e_{260} &= (950976\,c_{16} + 80064\,d_1 - 9216\,e_1 + 768\,e_2 - 395085\,n_v - 123264\,c_{16}\,v \\
&\quad - 5184\,d_1\,v + 147629\,n_v\,v + 4032\,c_{16}\,v^2 - 22306\,n_v\,v^2 + 1698\,n_v\,v^3 \\
&\quad - 65\,n_v\,v^4 + n_v\,v^5)/384 \\
e_{261} &= 6093\,c_{16} + 546\,d_1 - 120\,e_1 + 14\,e_2 + 12\,e_3 - (3219\,n_v)/2 - 935\,c_{16}\,v - 43\,d_1\,v \\
&\quad + (18291\,n_v\,v)/32 + 51\,c_{16}\,v^2 + d_1\,v^2 - (11951\,n_v\,v^2)/144 - c_{16}\,v^3 \\
&\quad + (49\,n_v\,v^3)/8 - (11\,n_v\,v^4)/48 + (n_v\,v^5)/288 \\
e_{262} &= 14865\,c_{16} + 1164\,d_1 - 168\,e_1 + 33\,e_2 + 36\,e_3 - 6\,e_4 - (3953\,n_v)/2 \\
&\quad - (6349\,c_{16}\,v)/2 - 111\,d_1\,v + (21293\,n_v\,v)/32 + (1633\,c_{16}\,v^2)/6 + 3\,d_1\,v^2 \\
&\quad - (2213\,n_v\,v^2)/24 - (65\,c_{16}\,v^3)/6 + (941\,n_v\,v^3)/144 + (c_{16}\,v^4)/6 \\
&\quad - (17\,n_v\,v^4)/72 + (n_v\,v^5)/288 \\
e_{263} &= (16821\,c_{16})/2 + 747\,d_1 - 108\,e_1 + 26\,e_2 + 30\,e_3 - 6\,e_4 - (30511\,n_v)/16 \\
&\quad - (2613\,c_{16}\,v)/2 - 52\,d_1\,v + (62537\,n_v\,v)/96 + (147\,c_{16}\,v^2)/2 + d_1\,v^2 \\
&\quad - (4381\,n_v\,v^2)/48 - (3\,c_{16}\,v^3)/2 + (469\,n_v\,v^3)/72 - (17\,n_v\,v^4)/72 \\
&\quad + (n_v\,v^5)/288
\end{aligned}
$$

$$\begin{aligned}
e_{264} &= 25314\, c_{16} + 2652\, d_1 - 240\, e_1 + 78\, e_2 + 72\, e_3 - 24\, e_4 - 36\, e_5 - (175917\, n_v)/32 \\
&\quad - 3936\, c_{16}\, v - 208\, d_1\, v + (180269\, n_v\, v)/96 + 214\, c_{16}\, v^2 + 4\, d_1\, v^2 \\
&\quad - (12683\, n_v\, v^2)/48 - 4\, c_{16}\, v^3 + (913\, n_v\, v^3)/48 - (67\, n_v\, v^4)/96 + (n_v\, v^5)/96 \\
e_{265} &= (2167488\, c_{16} + 199872\, d_1 - 27648\, e_1 + 8064\, e_2 + 9216\, e_3 - 2304\, e_4 \\
&\quad - 519219\, n_v - 305856\, c_{16}\, v - 10944\, d_1\, v + 178257\, n_v\, v + 14016\, c_{16}\, v^2 \\
&\quad - 25210\, n_v\, v^2 - 192\, c_{16}\, v^3 + 1822\, n_v\, v^3 - 67\, n_v\, v^4 + n_v\, v^5)/384 \\
e_{266} &= 4137\, c_{16} + 393\, d_1 - 27\, e_1 + 10\, e_2 + 9\, e_3 - 3\, e_4 - 9\, e_5 - (7173\, n_v)/8 \\
&\quad - (6055\, c_{16}\, v)/8 - 34\, d_1\, v + (1237\, n_v\, v)/4 + (1447\, c_{16}\, v^2)/24 + d_1\, v^2 \\
&\quad - (25229\, n_v\, v^2)/576 - (59\, c_{16}\, v^3)/24 + (1823\, n_v\, v^3)/576 + (c_{16}\, v^4)/24 \\
&\quad - (67\, n_v\, v^4)/576 + (n_v\, v^5)/576 \\
e_{267} &= 38628\, c_{16} + 4110\, d_1 - 420\, e_1 + 135\, e_2 + 132\, e_3 - 42\, e_4 - 36\, e_5 \\
&\quad - (171401\, n_v)/16 - 5220\, c_{16}\, v - 288\, d_1\, v + (59187\, n_v\, v)/16 + 244\, c_{16}\, v^2 \\
&\quad + 6\, d_1\, v^2 - (12593\, n_v\, v^2)/24 - 4\, c_{16}\, v^3 + (911\, n_v\, v^3)/24 - (67\, n_v\, v^4)/48 \\
&\quad + (n_v\, v^5)/48 \\
e_{268} &= (1627296\, c_{16} + 153600\, d_1 - 18432\, e_1 + 5312\, e_2 + 4992\, e_3 - 1536\, e_4 - 2304\, e_5 \\
&\quad - 501911\, n_v - 204960\, c_{16}\, v - 7488\, d_1\, v + 175077\, n_v\, v + 8160\, c_{16}\, v^2 \\
&\quad - 25014\, n_v\, v^2 - 96\, c_{16}\, v^3 + 1818\, n_v\, v^3 - 67\, n_v\, v^4 + n_v\, v^5)/64 \\
e_{269} &= 12102\, c_{16} + 1200\, d_1 - 150\, e_1 + 49\, e_2 + 54\, e_3 - 15\, e_4 - 9\, e_5 \\
&\quad - (505255\, n_v)/128 - 1350\, c_{16}\, v - 54\, d_1\, v + (175525\, n_v\, v)/128 + 39\, c_{16}\, v^2 \\
&\quad - (12515\, n_v\, v^2)/64 + (909\, n_v\, v^3)/64 - (67\, n_v\, v^4)/128 + (n_v\, v^5)/128 \\
e_{270} &= ((-13 + v)\, (311040\, c_{16} + 13824\, d_1 - 344970\, n_v - 20736\, c_{16}\, v + 134583\, n_v\, v \\
&\quad - 21014\, n_v\, v^2 + 1640\, n_v\, v^3 - 64\, n_v\, v^4 + n_v\, v^5))/27648 \\
e_{271} &= -14304\, c_{16} - 1032\, d_1 + 192\, e_1 - 24\, e_2 - 24\, e_3 + (398797\, n_v)/64 \\
&\quad + 2194\, c_{16}\, v + 70\, d_1\, v - (998075\, n_v\, v)/384 - 114\, c_{16}\, v^2 - d_1\, v^2 \\
&\quad + (176591\, n_v\, v^2)/384 + 2\, c_{16}\, v^3 - (25343\, n_v\, v^3)/576 + (115\, n_v\, v^4)/48 \\
&\quad - (9\, n_v\, v^5)/128 + (n_v\, v^6)/1152 \\
e_{272} &= (-11019\, c_{16})/2 - 357\, d_1 + 48\, e_1 - 6\, e_2 - 6\, e_3 + (376853\, n_v)/128 \\
&\quad + (1625\, c_{16}\, v)/2 + 21\, d_1\, v - (963395\, n_v\, v)/768 - (75\, c_{16}\, v^2)/2 \\
&\quad + (519301\, n_v\, v^2)/2304 + (c_{16}\, v^3)/2 - (8369\, n_v\, v^3)/384 + (43\, n_v\, v^4)/36 \\
&\quad - (9\, n_v\, v^5)/256 + (n_v\, v^6)/2304
\end{aligned}$$

$$
\begin{aligned}
e_{273} &= -5463\,c_{16} - 408\,d_1 + 56\,e_1 - 11\,e_2 - 12\,e_3 + 2\,e_4 + (29795\,n_v)/24 \\
&+ (2223\,c_{16}\,v)/2 + 37\,d_1\,v - (13111\,n_v\,v)/27 - (1657\,c_{16}\,v^2)/18 - d_1\,v^2 \\
&+ (7789\,n_v\,v^2)/96 + (65\,c_{16}\,v^3)/18 - (14335\,n_v\,v^3)/1944 - (c_{16}\,v^4)/18 \\
&+ (497\,n_v\,v^4)/1296 - (7\,n_v\,v^5)/648 + (n_v\,v^6)/7776 \\
e_{274} &= -10669\,c_{16} - 895\,d_1 + 96\,e_1 - 24\,e_2 - 24\,e_3 + 6\,e_4 + 6\,e_5 \\
&+ (169037\,n_v)/64 + (3979\,c_{16}\,v)/2 + (230\,d_1\,v)/3 - (1195469\,n_v\,v)/1152 \\
&- (2575\,c_{16}\,v^2)/18 - (5\,d_1\,v^2)/3 + (603601\,n_v\,v^2)/3456 + (83\,c_{16}\,v^3)/18 \\
&- (27691\,n_v\,v^3)/1728 - (c_{16}\,v^4)/18 + (91\,n_v\,v^4)/108 - (83\,n_v\,v^5)/3456 \\
&+ (n_v\,v^6)/3456 \\
e_{275} &= -47037\,c_{16} - 4086\,d_1 + 408\,e_1 - 111\,e_2 - 108\,e_3 + 30\,e_4 + 36\,e_5 \\
&+ (490859\,n_v)/32 + (15661\,c_{16}\,v)/2 + 319\,d_1\,v - (1172041\,n_v\,v)/192 \\
&- (3025\,c_{16}\,v^2)/6 - 7\,d_1\,v^2 + (66357\,n_v\,v^2)/64 + (89\,c_{16}\,v^3)/6 \\
&- (9187\,n_v\,v^3)/96 - (c_{16}\,v^4)/6 + (727\,n_v\,v^4)/144 - (83\,n_v\,v^5)/576 \\
&+ (n_v\,v^6)/576 \\
e_{276} &= -57972\,c_{16} - 5067\,d_1 + 552\,e_1 - 153\,e_2 - 156\,e_3 + 42\,e_4 + 36\,e_5 \\
&+ (1440077\,n_v)/64 + (16645\,c_{16}\,v)/2 + (657\,d_1\,v)/2 - (1154749\,n_v\,v)/128 \\
&- 403\,c_{16}\,v^2 - (9\,d_1\,v^2)/2 + (591977\,n_v\,v^2)/384 + (13\,c_{16}\,v^3)/2 \\
&- (27443\,n_v\,v^3)/192 + (121\,n_v\,v^4)/16 - (83\,n_v\,v^5)/384 + (n_v\,v^6)/384 \\
e_{277} &= (-24553\,c_{16})/4 - 513\,d_1 + 56\,e_1 - 15\,e_2 - 14\,e_3 + 4\,e_4 + 6\,e_5 \\
&+ (4196627\,n_v)/1536 + (6549\,c_{16}\,v)/8 + 27\,d_1\,v - (1134081\,n_v\,v)/1024 \\
&- (135\,c_{16}\,v^2)/4 + (195351\,n_v\,v^2)/1024 + (3\,c_{16}\,v^3)/8 - (27317\,n_v\,v^3)/1536 \\
&+ (725\,n_v\,v^4)/768 - (83\,n_v\,v^5)/3072 + (n_v\,v^6)/3072 \\
e_{278} &= 3273\,c_{16} + 207\,d_1 - 32\,e_1 + 4\,e_2 + 4\,e_3 - (790321\,n_v)/384 \\
&- (1045\,c_{16}\,v)/2 - (85\,d_1\,v)/6 + (6688423\,n_v\,v)/6912 + 28\,c_{16}\,v^2 + (d_1\,v^2)/6 \\
&- (683659\,n_v\,v^2)/3456 - (c_{16}\,v^3)/2 + (1413707\,n_v\,v^3)/62208 \\
&- (49211\,n_v\,v^4)/31104 + (461\,n_v\,v^5)/6912 - (49\,n_v\,v^6)/31104 + (n_v\,v^7)/62208 \\
e_{279} &= 37047\,c_{16} + 2865\,d_1 - 288\,e_1 + 72\,e_2 + 72\,e_3 - 18\,e_4 - 18\,e_5 \\
&- (1001563\,n_v)/64 - (13047\,c_{16}\,v)/2 - 230\,d_1\,v + (8049425\,n_v\,v)/1152 \\
&+ (2665\,c_{16}\,v^2)/6 + 5\,d_1\,v^2 - (98435\,n_v\,v^2)/72 - (83\,c_{16}\,v^3)/6 \\
&+ (1568311\,n_v\,v^3)/10368 + (c_{16}\,v^4)/6 - (52859\,n_v\,v^4)/5184 \\
&+ (1445\,n_v\,v^5)/3456 - (25\,n_v\,v^6)/2592 + (n_v\,v^7)/10368
\end{aligned}
$$

$$\begin{aligned}
e_{280} &= 34041\,c_{16} + (5307\,d_1)/2 - 264\,e_1 + 69\,e_2 + 69\,e_3 - 18\,e_4 - 18\,e_5 \\
&\quad - (8766103\,n_v)/512 - (42969\,c_{16}\,v)/8 - (375\,d_1\,v)/2 + (7899109\,n_v\,v)/1024 \\
&\quad + (7243\,c_{16}\,v^2)/24 + 3\,d_1\,v^2 - (194561\,n_v\,v^2)/128 - (161\,c_{16}\,v^3)/24 \\
&\quad + (519341\,n_v\,v^3)/3072 + (c_{16}\,v^4)/24 - (52699\,n_v\,v^4)/4608 + (4331\,n_v\,v^5)/9216 \\
&\quad - (25\,n_v\,v^6)/2304 + (n_v\,v^7)/9216 \\
e_{281} &= (-43347\,c_{16})/2 - 1545\,d_1 + 144\,e_1 - 36\,e_2 - 36\,e_3 + 9\,e_4 + 9\,e_5 \\
&\quad + (404907\,n_v)/32 + (30549\,c_{16}\,v)/8 + (245\,d_1\,v)/2 - (9579917\,n_v\,v)/1536 \\
&\quad - (3055\,c_{16}\,v^2)/12 - (5\,d_1\,v^2)/2 + (37932923\,n_v\,v^2)/27648 + (181\,c_{16}\,v^3)/24 \\
&\quad - (2422781\,n_v\,v^3)/13824 - (c_{16}\,v^4)/12 + (3531139\,n_v\,v^4)/248832 \\
&\quad - (92675\,n_v\,v^5)/124416 + (2051\,n_v\,v^6)/82944 - (59\,n_v\,v^7)/124416 \\
&\quad + (n_v\,v^8)/248832 \\
e_{282} &= (5563\,c_{16})/2 + 185\,d_1 - 16\,e_1 + 4\,e_2 + 4\,e_3 - e_4 - e_5 - (1709173\,n_v)/864 \\
&\quad - (4109\,c_{16}\,v)/8 - (46\,d_1\,v)/3 + (655601021\,n_v\,v)/622080 + (2611\,c_{16}\,v^2)/72 \\
&\quad + (d_1\,v^2)/3 - (949373699\,n_v\,v^2)/3732480 - (83\,c_{16}\,v^3)/72 \\
&\quad + (407365571\,n_v\,v^3)/11197440 + (c_{16}\,v^4)/72 - (113957779\,n_v\,v^4)/33592320 \\
&\quad + (7176053\,n_v\,v^5)/33592320 - (304873\,n_v\,v^6)/33592320 + (8419\,n_v\,v^7)/33592320 \\
&\quad - (137\,n_v\,v^8)/33592320 + (n_v\,v^9)/33592320
\end{aligned}$$

# Appendix C

# Configurations of four or fewer blocks

This is a complete listing of all configurations of four or fewer blocks that can occur in triple systems.

The first column is the configurations's name. The second column is $b(X)$, the number of blocks. The third column is $p(X)$, the number of points. The fourth column gives the blocks of the configuration with its canonical labelling. The fifth column indicates the degrees of the points in the order $0, 1, \ldots, p(X)-1$; generating configurations are starred. The sixth column gives order of the full automorphism group.

| $X$ | $b(X)$ | $p(X)$ | blocks | degrees | $\lvert\mathrm{Aut}(X)\rvert$ |
|---|---:|---:|---|---|---:|
| $A_0$ | 1 | 3 | 012 | 111 | 6 |
| $A_2$ | 2 | 5 | 012 034 | 21111 | 8 |
| $A_1$ | 2 | 6 | 012 345 | 111111 | 72 |
| $A_4$ | 2 | 3 | 012 012 | 222* | 12 |
| $A_3$ | 2 | 4 | 012 013 | 2211 | 4 |

| $X$ | $b(X)$ | $p(X)$ | blocks | degrees | $\|\mathrm{Aut}(X)\|$ |
|---|---|---|---|---|---|
| $B_5$ | 3 | 6 | 012 034 135 | 221211 | 6 |
| $B_3$ | 3 | 7 | 012 034 056 | 3111111 | 48 |
| $B_4$ | 3 | 7 | 012 034 156 | 2211111 | 8 |
| $B_2$ | 3 | 8 | 012 034 567 | 21111111 | 48 |
| $B_1$ | 3 | 9 | 012 345 678 | 111111111 | 1296 |
| $B_6$ | 3 | 4 | 012 013 023 | 3222* | 6 |
| $B_{13}$ | 3 | 5 | 012 012 034 | 32211 | 8 |
| $B_7$ | 3 | 5 | 012 013 024 | 32211 | 2 |
| $B_8$ | 3 | 5 | 012 013 234 | 22221 | 4 |
| $B_{12}$ | 3 | 6 | 012 012 345 | 222111 | 72 |
| $B_9$ | 3 | 6 | 012 013 045 | 321111 | 4 |
| $B_{10}$ | 3 | 6 | 012 013 245 | 222111 | 4 |
| $B_{11}$ | 3 | 7 | 012 013 456 | 2211111 | 24 |
| $B_{16}$ | 3 | 3 | 012 012 012 | 333* | 36 |
| $B_{14}$ | 3 | 4 | 012 012 013 | 3321 | 4 |
| $B_{15}$ | 3 | 5 | 012 013 014 | 33111 | 12 |

| $X$ | $b(X)$ | $p(X)$ | blocks | degrees | $\|\mathrm{Aut}(X)\|$ |
|---|---|---|---|---|---|
| $C_{16}$ | 4 | 6 | 012 034 135 245 | 222222* | 24 |
| $C_{15}$ | 4 | 7 | 012 034 135 236 | 2223111 | 6 |
| $C_{14}$ | 4 | 7 | 012 034 135 246 | 2222211 | 4 |
| $C_{11}$ | 4 | 8 | 012 034 135 067 | 32121111 | 4 |
| $C_{12}$ | 4 | 8 | 012 034 135 267 | 22221111 | 4 |
| $C_{10}$ | 4 | 8 | 012 034 156 357 | 22121211 | 8 |
| $C_7$ | 4 | 9 | 012 034 056 078 | 411111111 | 384 |
| $C_8$ | 4 | 9 | 012 034 056 178 | 321111111 | 16 |
| $C_6$ | 4 | 9 | 012 034 135 678 | 221211111 | 36 |
| $C_{13}$ | 4 | 9 | 012 034 156 278 | 222111111 | 48 |
| $C_9$ | 4 | 9 | 012 034 156 378 | 221211111 | 8 |
| $C_4$ | 4 | 10 | 012 034 056 789 | 3111111111 | 288 |
| $C_5$ | 4 | 10 | 012 034 156 789 | 2211111111 | 48 |
| $C_3$ | 4 | 10 | 012 034 567 589 | 2111121111 | 128 |
| $C_2$ | 4 | 11 | 012 034 567 89a | 21111111111 | 576 |
| $C_1$ | 4 | 12 | 012 345 678 9ab | 111111111111 | 31104 |

| $X$ | $b(X)$ | $p(X)$ | blocks | degrees | $\lvert\mathrm{Aut}(X)\rvert$ |
|---|---|---|---|---|---|
| $C_{17}$ | 4 | 4 | 012 013 023 123 | 3333* | 24 |
| $C_{18}$ | 4 | 5 | 012 012 034 034 | 42222* | 32 |
| $C_{19}$ | 4 | 5 | 012 012 034 134 | 33222* | 8 |
| $C_{20}$ | 4 | 5 | 012 013 024 034 | 42222* | 8 |
| $C_{21}$ | 4 | 5 | 012 013 024 134 | 33222* | 2 |
| $C_{22}$ | 4 | 6 | 012 012 345 345 | 222222* | 288 |
| $C_{23}$ | 4 | 6 | 012 013 245 345 | 222222* | 16 |
| $C_{24}$ | 4 | 5 | 012 013 023 124 | 33321 | 2 |
| $C_{25}$ | 4 | 6 | 012 012 034 035 | 422211 | 8 |
| $C_{26}$ | 4 | 6 | 012 012 034 135 | 332211 | 4 |
| $C_{27}$ | 4 | 6 | 012 012 034 345 | 322221 | 8 |
| $C_{28}$ | 4 | 6 | 012 013 023 045 | 422211 | 12 |
| $C_{29}$ | 4 | 6 | 012 013 023 145 | 332211 | 4 |
| $C_{30}$ | 4 | 6 | 012 013 024 035 | 422211 | 2 |
| $C_{31}$ | 4 | 6 | 012 013 024 125 | 333111 | 6 |
| $C_{32}$ | 4 | 6 | 012 013 024 135 | 332211 | 2 |
| $C_{33}$ | 4 | 6 | 012 013 024 235 | 323211 | 1 |
| $C_{34}$ | 4 | 6 | 012 013 024 345 | 322221 | 2 |
| $C_{35}$ | 4 | 6 | 012 013 234 045 | 322221 | 2 |
| $C_{36}$ | 4 | 6 | 012 013 234 235 | 223311 | 8 |
| $C_{37}$ | 4 | 6 | 012 013 234 245 | 223221 | 2 |
| $C_{38}$ | 4 | 7 | 012 012 034 056 | 4221111 | 32 |
| $C_{39}$ | 4 | 7 | 012 012 034 156 | 3321111 | 16 |
| $C_{40}$ | 4 | 7 | 012 012 034 356 | 3222111 | 8 |
| $C_{41}$ | 4 | 7 | 012 012 345 346 | 2222211 | 48 |
| $C_{42}$ | 4 | 7 | 012 013 023 456 | 3222111 | 36 |
| $C_{43}$ | 4 | 7 | 012 013 024 056 | 4221111 | 4 |
| $C_{44}$ | 4 | 7 | 012 013 024 156 | 3321111 | 2 |
| $C_{45}$ | 4 | 7 | 012 013 024 356 | 3222111 | 2 |
| $C_{46}$ | 4 | 7 | 012 013 045 046 | 4211211 | 8 |
| $C_{47}$ | 4 | 7 | 012 013 045 146 | 3311211 | 4 |
| $C_{48}$ | 4 | 7 | 012 013 045 246 | 3221211 | 1 |
| $C_{49}$ | 4 | 7 | 012 013 045 456 | 3211221 | 4 |
| $C_{50}$ | 4 | 7 | 012 013 234 056 | 3222111 | 4 |
| $C_{51}$ | 4 | 7 | 012 013 234 256 | 2232111 | 4 |
| $C_{52}$ | 4 | 7 | 012 013 234 456 | 2222211 | 8 |
| $C_{53}$ | 4 | 7 | 012 013 245 346 | 2222211 | 4 |
| $C_{54}$ | 4 | 7 | 012 013 245 456 | 2221221 | 8 |

| $X$ | $b(X)$ | $p(X)$ | blocks | degrees | $\|\mathrm{Aut}(X)\|$ |
|---|---|---|---|---|---|
| $C_{55}$ | 4 | 8 | 012 012 034 567 | 32211111 | 48 |
| $C_{56}$ | 4 | 8 | 012 012 345 367 | 22221111 | 96 |
| $C_{57}$ | 4 | 8 | 012 013 024 567 | 32211111 | 12 |
| $C_{58}$ | 4 | 8 | 012 013 045 067 | 42111111 | 16 |
| $C_{59}$ | 4 | 8 | 012 013 045 167 | 33111111 | 16 |
| $C_{60}$ | 4 | 8 | 012 013 045 267 | 32211111 | 4 |
| $C_{61}$ | 4 | 8 | 012 013 045 467 | 32112111 | 4 |
| $C_{62}$ | 4 | 8 | 012 013 234 567 | 22221111 | 24 |
| $C_{63}$ | 4 | 8 | 012 013 245 267 | 22311111 | 16 |
| $C_{64}$ | 4 | 8 | 012 013 245 367 | 22221111 | 16 |
| $C_{65}$ | 4 | 8 | 012 013 245 467 | 22212111 | 4 |
| $C_{66}$ | 4 | 8 | 012 013 456 457 | 22112211 | 32 |
| $C_{67}$ | 4 | 9 | 012 012 345 678 | 222111111 | 864 |
| $C_{68}$ | 4 | 9 | 012 013 045 678 | 321111111 | 24 |
| $C_{69}$ | 4 | 9 | 012 013 245 678 | 222111111 | 24 |
| $C_{70}$ | 4 | 9 | 012 013 456 478 | 221121111 | 32 |
| $C_{71}$ | 4 | 10 | 012 013 456 789 | 2211111111 | 288 |
| $C_{72}$ | 4 | 4 | 012 012 013 023 | 4332* | 4 |
| $C_{73}$ | 4 | 5 | 012 013 014 234 | 33222* | 12 |
| $C_{74}$ | 4 | 5 | 012 012 012 034 | 43311 | 24 |
| $C_{75}$ | 4 | 5 | 012 012 013 024 | 43311 | 4 |
| $C_{76}$ | 4 | 5 | 012 012 013 034 | 43221 | 2 |
| $C_{77}$ | 4 | 5 | 012 012 013 234 | 33321 | 4 |
| $C_{78}$ | 4 | 5 | 012 013 014 023 | 43221 | 2 |
| $C_{79}$ | 4 | 6 | 012 012 012 345 | 333111 | 216 |
| $C_{80}$ | 4 | 6 | 012 012 013 045 | 432111 | 4 |
| $C_{81}$ | 4 | 6 | 012 012 013 245 | 333111 | 8 |
| $C_{82}$ | 4 | 6 | 012 012 013 345 | 332211 | 8 |
| $C_{83}$ | 4 | 6 | 012 013 014 025 | 432111 | 2 |
| $C_{84}$ | 4 | 6 | 012 013 014 235 | 332211 | 4 |
| $C_{85}$ | 4 | 7 | 012 012 013 456 | 3321111 | 24 |
| $C_{86}$ | 4 | 7 | 012 013 014 056 | 4311111 | 12 |
| $C_{87}$ | 4 | 7 | 012 013 014 256 | 3321111 | 8 |
| $C_{88}$ | 4 | 8 | 012 013 014 567 | 33111111 | 72 |
| $C_{89}$ | 4 | 3 | 012 012 012 012 | 444* | 144 |
| $C_{90}$ | 4 | 4 | 012 012 013 013 | 4422* | 16 |
| $C_{91}$ | 4 | 4 | 012 012 012 013 | 4431 | 12 |
| $C_{92}$ | 4 | 5 | 012 012 013 014 | 44211 | 8 |
| $C_{93}$ | 4 | 6 | 012 013 014 015 | 441111 | 48 |

# Appendix D

# Formulae for 4-block configurations

The formulae for the number of occurrences of $n$-block configurations in triple systems, $n = 1, 2, 3, 4$. See section 7.3. There are 15 generators: $A_4$, $B_6$, $B_{16}$, $C_{16}$, $C_{17}$, $C_{18}$, $C_{19}$, $C_{20}$, $C_{21}$, $C_{22}$, $C_{23}$, $C_{72}$, $C_{73}$, $C_{89}$ and $C_{90}$. We write $x_i^*$ for $|\mathrm{Aut}(X_i)|\, x_i$, where $|\mathrm{Aut}(X_i)|$ can be obtained from Appendix C.

$$a_0^* = \lambda v(v-1)$$

$$\begin{aligned}
a_1^* &= -6a_4^* + \lambda(-1+v)v(-6+\lambda(27-10v+v^2)) \\
a_2^* &= 2a_4^* + \lambda(2+\lambda(-5+v))(-1+v)v \\
a_3^* &= -a_4^* + (-1+\lambda)\lambda(-1+v)v
\end{aligned}$$

$$\begin{aligned}
b_1^* &= 288b_{16}^* - 432b_6^* - 18a_4^*(-48+\lambda(108+(-19+v)v)) \\
&\quad + \lambda(-1+v)v(288-18\lambda(108+(-19+v)v) \\
&\quad + \lambda^2(3672+v(-1701+v(325+(-29+v)v)))) \\
b_2^* &= -48b_{16}^* + 72b_6^* + 2a_4^*(-72+\lambda(141+(-22+v)v)) \\
&\quad + \lambda(-1+v)v(-48+2\lambda(141 \\
&\quad + (-22+v)v) + \lambda^2(-9+v)(47+(-12+v)v)) \\
b_3^* &= 16b_{16}^* - 8b_6^* + 6a_4^*(8+\lambda(-9+v)) \\
&\quad + \lambda(-1+v)v(16+\lambda(6(-9+v)+\lambda(53+(-14+v)v))) \\
b_4^* &= 8b_{16}^* - 16b_6^* + a_4^*(24+\lambda(-42+4v)) \\
&\quad + \lambda(-1+v)v(8+\lambda(-42+4v+\lambda(55+(-14+v)v))) \\
b_5^* &= -b_{16}^* + 5b_6^* + a_4^*(-3+6\lambda) + \lambda(-1+\lambda(6+\lambda(-8+v)))(-1+v)v
\end{aligned}$$

$$
\begin{aligned}
b_7^* &= b_{16}^* - b_6^* + a_4^*(3-2\lambda) + (-1+\lambda)^2\lambda(-1+v)v \\
b_8^* &= -2b_6^* + \lambda(-a_4^* + (-1+\lambda)\lambda(-1+v)v) \\
b_9^* &= -4b_{16}^* + 2b_6^* - a_4^*(12+\lambda(-11+v)) \\
&\quad + (-1+\lambda)\lambda(4+\lambda(-7+v))(-1+v)v \\
b_{10}^* &= -2b_{16}^* + 4b_6^* - a_4^*(6+\lambda(-9+v)) \\
&\quad + (-1+\lambda)\lambda(2+\lambda(-7+v))(-1+v)v \\
b_{11}^* &= 12b_{16}^* - 12b_6^* + (-1+\lambda)\lambda(-1+v)v(-12+\lambda(48+(-13+v)v)) \\
&\quad + a_4^*(36-\lambda(60+(-13+v)v)) \\
b_{12}^* &= -6b_{16}^* + a_4^*(-12+\lambda(27+(-10+v)v)) \\
b_{13}^* &= 2b_{16}^* + a_4^*(4+\lambda(-5+v)) \\
b_{14}^* &= -b_{16}^* + a_4^*(-2+\lambda) \\
b_{15}^* &= 2b_{16}^* + (-2+\lambda)(-3a_4^* + (-1+\lambda)\lambda(-1+v)v)
\end{aligned}
$$

$$
\begin{aligned}
c_1^* &= 108(1152b_6^* + 12c_{16}^* + 192c_{17}^* - 27c_{18}^* - 144c_{19}^* - 108c_{20}^* \\
&\quad - 576c_{21}^* + c_{22}^* + 18c_{23}^* + 864c_{72}^* - 96c_{73}^* - 54c_{89}^* + 162c_{90}^*) \\
&\quad + 1152b_{16}^*(-72+\lambda(243+(-28+v)v)) \\
&\quad - 36a_4^*(4464 - 6\lambda(4284+v(-485+17v)) \\
&\quad + \lambda^2(30942+v(-8019+v(883+(-47+v)v)))) \\
&\quad + \lambda(-1728b_6^*(243+(-28+v)v) \\
&\quad + (-1+v)v(-29808+36\lambda(9180+v(-1007+35v)) \\
&\quad - 36\lambda^2(31374+v(-8019+v(883+(-47+v)v))) \\
&\quad + \lambda^3(1323702+v(-637173+v(143073+v(-18334+v(1380+(-57+v)v))))))) \\
c_2^* &= -12(1152b_6^* + 12c_{16}^* + 192c_{17}^* - 27c_{18}^* - 144c_{19}^* - 108c_{20}^* - 576c_{21}^* + c_{22}^* \\
&\quad + 18c_{23}^* + 864c_{72}^* - 96c_{73}^* - 54c_{89}^* + 162c_{90}^*) \\
&\quad - 96b_{16}^*(-96+\lambda(294+(-31+v)v)) \\
&\quad + 2a_4^*(8928 - 12\lambda(3924+v(-409+13v)) \\
&\quad + \lambda^2(52164+v(-12276+v(1207+(-56+v)v)))) \\
&\quad + \lambda(144b_6^*(294+(-31+v)v) + (-1+v)v(3312 - 36\lambda(940+v(-95+3v)) \\
&\quad + 2\lambda^2(53028+v(-12276+v(1207+(-56+v)v))) \\
&\quad + \lambda^3(-110358+v(46035+v(-8648+v(872+(-46+v)v))))))
\end{aligned}
$$

$$
\begin{aligned}
c_3^* &= 4(416b_6^* + 4c_{16}^* + 64c_{17}^* - 16c_{18}^* - 72c_{19}^* - 40c_{20}^* - 224c_{21}^* + c_{22}^* + 10c_{23}^* \\
&+ 272c_{72}^* - 32c_{73}^* - 12c_{89}^* + 54c_{90}^*) - 64b_{16}^*(16 + 3\lambda(-15 + v)) \\
&+ 4a_4^*(-544 + 2\lambda(1277 + (-98 + v)v) + \lambda^2(-2635 + v(511 + (-37 + v)v))) \\
&+ \lambda(288b_6^*(-15 + v) + (-1 + v)v(-416 + 4\lambda(953 + (-74 + v)v) \\
&+ 4\lambda^2(-2707 + v(511 + (-37 + v)v)) \\
&+ \lambda^3(10105 + v(-3476 + v(510 + (-36 + v)v)))))) \\
c_4^* &= -12(192b_{16}^* - 192b_6^* - 32c_{17}^* + 3c_{18}^* + 2(6c_{19}^* + 6c_{20}^* + 24c_{21}^* - 84c_{72}^* \\
&+ 4c_{73}^* + 9c_{89}^* - 18c_{90}^*)) + 6a_4^*(-672 + 4\lambda(621 + v(-65 + 2v)) \\
&+ \lambda^2(-1998 + v(423 + (-34 + v)v))) + \lambda(-8b_6^*(552 + (-49 + v)v) \\
&+ 16b_{16}^*(282 + (-31 + v)v) + (-1 + v)v(-720 + \lambda(5268 + 4v(-133 + 4v) \\
&+ 6\lambda(-2022 + v(423 + (-34 + v)v)) \\
&+ \lambda^2(9663 + v(-3416 + v(508 + (-36 + v)v)))))) \\
c_5^* &= -12(80b_{16}^* - 136b_6^* - 2c_{16}^* - 24c_{17}^* + 2c_{18}^* + 15c_{19}^* \\
&+ 2(7c_{20}^* + 38c_{21}^* - c_{23}^* - 53c_{72}^* + 7c_{73}^* + 3c_{89}^* - 9c_{90}^*)) \\
&+ 4a_4^*(-456 + 3\lambda(797 + 2(-36 + v)v) + \lambda^2(-2526 + v(495 + (-37 + v)v))) \\
&+ \lambda(8b_{16}^*(363 + (-34 + v)v) - 16b_6^*(309 + (-31 + v)v) \\
&+ (-1 + v)v(-336 + 4\lambda(855 + 2(-37 + v)v) \\
&+ 4\lambda^2(-2571 + v(495 + (-37 + v)v)) \\
&+ \lambda^3(9981 + v(-3460 + v(510 + (-36 + v)v)))))) \\
c_6^* &= 6(12b_{16}^* - 36b_6^* - c_{16}^* - 8c_{17}^* + 3c_{19}^* + 3c_{20}^* + 24c_{21}^* - 30c_{72}^* + 6c_{73}^* + c_{89}^* \\
&- 3c_{90}^*) + 3a_4^*(44 + \lambda(-270 - (-19 + v)v + 2\lambda(156 + (-22 + v)v))) \\
&+ \lambda((-b_{16}^*)(252 + (-19 + v)v) + b_6^*(756 + 5(-19 + v)v) \\
&+ (-1 + v)v(24 + \lambda(-288 - (-19 + v)v + 6\lambda(159 + (-22 + v)v) \\
&+ \lambda^2(-12 + v)(80 + (-15 + v)v)))) \\
c_7^* &= 12(64b_{16}^* - 32b_6^* + c_{18}^* + 4c_{20}^* - 32c_{72}^* + 6c_{89}^* - 12c_{90}^*) \\
&+ 12a_4^*(112 + \lambda(-242 + 18v + \lambda(133 + (-22 + v)v))) \\
&+ \lambda(64b_{16}^*(-13 + v) - 32b_6^*(-13 + v) \\
&+ (-1 + v)v(240 + \lambda(-1036 + 76v + 12\lambda(133 + (-22 + v)v) \\
&+ \lambda^2(-905 + v(259 + (-27 + v)v))))) \\
c_8^* &= 4(64b_{16}^* - 80b_6^* - 16c_{17}^* + c_{18}^* + 6c_{19}^* + 4c_{20}^* + 24c_{21}^* - 68c_{72}^* + 4c_{73}^* \\
&+ 6c_{89}^* - 12c_{90}^*) + 4a_4^*(112 + \lambda(-388 + 26v + \lambda(307 + v(-47 + 2v)))) \\
&+ \lambda(16b_{16}^*(-29 + 2v) - 8b_6^*(-74 + 5v) \\
&+ (-1 + v)v(80 + \lambda(-556 + 36v + 4\lambda(313 + v(-47 + 2v)) \\
&+ \lambda^2(-965 + v(263 + (-27 + v)v)))))
\end{aligned}
$$

$$
\begin{aligned}
c_9^* &= 2(44b_{16}^* - 100b_6^* - 2c_{16}^* - 16c_{17}^* + 2c_{18}^* + 15c_{19}^* + 14c_{20}^* + 72c_{21}^* - 3c_{23}^* \\
&\quad + 2(-35c_{72}^* + 6c_{73}^* + c_{89}^* - 6c_{90}^*)) \\
&\quad + 2a_4^*(94 + \lambda(-515 + 28v + \lambda(533 + 3(-25+v)v))) \\
&\quad + \lambda(b_6^*(560 - 32v) + 16b_{16}^*(-19+v) \\
&\quad + (-1+v)v(36 + \lambda(-378 + 20v + 2\lambda(548 + 3(-25+v)v) \\
&\quad + \lambda^2(-997 + v(265 + (-27+v)v)))))) \\
c_{10}^* &= -4b_{16}^* + 20b_6^* + c_{16}^* + 2c_{17}^* - 4c_{19}^* - 7c_{20}^* \\
&\quad + 2(-14c_{21}^* + c_{23}^* + 6c_{72}^* - 2c_{73}^* + c_{90}^*) + 2a_4^*(-5 + 52\lambda + \lambda^2(-58+4v)) \\
&\quad + \lambda(32b_{16}^* - 64b_6^* \\
&\quad + (-1+v)v(-2 + \lambda(38 + \lambda(8(-15+v) + \lambda(108 + (-19+v)v))))) \\
c_{11}^* &= 50b_6^* + 12c_{17}^* - 2(2c_{19}^* + c_{20}^* + 10c_{21}^* - 21c_{72}^* + 2c_{73}^* + c_{89}^* - 2c_{90}^*) \\
&\quad - b_{16}^*(20 + \lambda(-43+v)) + a_4^*(-34 - 3\lambda(-47+v) + 2\lambda^2(-62+5v)) \\
&\quad + \lambda(5b_6^*(-19+v) + (-1+v)v(-6 + \lambda(51 - v + \lambda(2(-64+5v) \\
&\quad + \lambda(104 + (-19+v)v))))) \\
c_{12}^* &= 2(11b_6^* + c_{16}^* + 2c_{17}^* - c_{19}^* - 2c_{20}^* - 14c_{21}^* + 9c_{72}^* - 4c_{73}^* + c_{90}^*) \\
&\quad + a_4^*(-10 + \lambda(87 + 8\lambda(-13+v) - 3v)) - b_{16}^*(4 + \lambda(-27+v)) \\
&\quad + \lambda(b_6^*(-87+5v) + (-1+v)v(-2 + \lambda(31 - v \\
&\quad + \lambda(-106 + 8v + \lambda(104 + (-19+v)v))))) \\
c_{13}^* &= -4(48b_6^* + 2c_{16}^* + 8c_{17}^* - 3c_{19}^* - 2(3c_{20}^* + 18c_{21}^* - 21c_{72}^* + 5c_{73}^* + c_{89}^* \\
&\quad - 3c_{90}^*)) + 24b_{16}^*(4 + \lambda(-13+v)) \\
&\quad + 2a_4^*(88 + 6\lambda(-83+6v) + 3\lambda^2(172 + (-25+v)v)) \\
&\quad + \lambda(-48b_6^*(-13+v) + (-1+v)v(32 \\
&\quad + \lambda(-348 + 24v + 6\lambda(174 + (-25+v)v) \\
&\quad + \lambda^2(-983 + v(265 + (-27+v)v))))) \\
c_{14}^* &= -c_{16}^* + c_{20}^* + 6c_{21}^* + 2c_{73}^* + \lambda(-2b_{16}^* + 10b_6^* + 2a_4^*(-3+5\lambda) \\
&\quad + \lambda(-2 + \lambda(10 + \lambda(-11+v)))(-1+v)v) \\
c_{15}^* &= -2c_{17}^* + 6c_{21}^* - 6c_{72}^* + 2c_{73}^* + 3b_6^*(-2+5\lambda) \\
&\quad + \lambda(-3b_{16}^* + 3a_4^*(-3+4\lambda) \\
&\quad + \lambda(-3 + \lambda(12 + \lambda(-12+v)))(-1+v)v)
\end{aligned}
$$

$$
\begin{aligned}
c^*_{24} &= -b^*_6 - c^*_{17} - c^*_{72} + b^*_6\lambda \\
c^*_{25} &= -c^*_{18} + 2(c^*_{72} + c^*_{90}) + 4b^*_{16}(-1+\lambda) + a^*_4(-1+\lambda)(8+\lambda(-7+v)) \\
c^*_{26} &= -c^*_{19} + 3c^*_{72} + c^*_{90} + b^*_{16}(-1+4\lambda) + a^*_4(-2+\lambda(9+\lambda(-7+v))) \\
c^*_{27} &= -c^*_{18} - 2c^*_{19} + (-1+\lambda)(2b^*_{16} + a^*_4(4+\lambda(-5+v))) \\
c^*_{28} &= 6c^*_{72} + b^*_6(6+\lambda(-7+v)) \\
c^*_{29} &= 2(c^*_{17} + 2c^*_{72}) + b^*_6(4+\lambda(-7+v)) \\
c^*_{30} &= 3b^*_6 - c^*_{20} + 3c^*_{72} + c^*_{90} + 2b^*_{16}(-1+\lambda) + a^*_4(-5+(8-3\lambda)\lambda) \\
&\quad + \lambda(-2b^*_6 + (-1+\lambda)^3(-1+v)v) \\
c^*_{31} &= 3b^*_6 + 2c^*_{17} + 3c^*_{72} - c^*_{89} + 3b^*_{16}(-2+\lambda) + a^*_4(-7-3(-3+\lambda)\lambda) \\
&\quad + \lambda(-3b^*_6 + (-1+\lambda)^3(-1+v)v) \\
c^*_{32} &= 2b^*_6 + c^*_{17} - c^*_{21} + 2c^*_{72} + c^*_{90} + 2b^*_{16}(-1+\lambda) \\
&\quad + a^*_4(-5+(8-3\lambda)\lambda) + \lambda(-2b^*_6 + (-1+\lambda)^3(-1+v)v) \\
c^*_{33} &= c^*_{17} - c^*_{21} + 3c^*_{72} - 3b^*_6(-1+\lambda) \\
&\quad + \lambda(b^*_{16} + a^*_4(3-2\lambda) + (-1+\lambda)^2\lambda(-1+v)v) \\
c^*_{34} &= -c^*_{20} - 2c^*_{21} + \lambda(b^*_{16} - b^*_6 + a^*_4(3-2\lambda) \\
&\quad + (-1+\lambda)^2\lambda(-1+v)v) \\
c^*_{35} &= -2c^*_{21} - c^*_{73} \\
&\quad + \lambda(-2b^*_6 + \lambda(-a^*_4 + (-1+\lambda)\lambda(-1+v)v)) \\
c^*_{36} &= 2c^*_{17} - c^*_{19} + 2c^*_{72} - 4b^*_6(-1+\lambda) - (-1+\lambda)\lambda(a^*_4 + \lambda v(-1+\lambda+v-\lambda v)) \\
c^*_{37} &= -c^*_{19} - 2c^*_{21} + (-1+\lambda)(-2b^*_6 + \lambda(-a^*_4 + (-1+\lambda)\lambda(-1+v)v)) \\
c^*_{38} &= 2(c^*_{18} - 4c^*_{72} + 2c^*_{89} - 4c^*_{90} + 2b^*_{16}(8+\lambda(-9+v))) \\
&\quad + a^*_4(40 + 10\lambda(-9+v) + \lambda^2(53+(-14+v)v)) \\
c^*_{39} &= 2(c^*_{19} - 6c^*_{72} + 2c^*_{89} - 2c^*_{90} + 2b^*_{16}(6+\lambda(-9+v))) \\
&\quad + a^*_4(24 + \lambda(-76 + 8v + \lambda(53+(-14+v)v))) \\
c^*_{40} &= 2(c^*_{18} + 2c^*_{19} - 4c^*_{72} - 2c^*_{90} + b^*_{16}(4+\lambda(-12+v))) \\
&\quad + a^*_4(16 + \lambda(-62 + 6v + \lambda(51+(-14+v)v))) \\
c^*_{41} &= 3c^*_{18} + 6c^*_{19} - c^*_{22} - 12b^*_{16}(-1+\lambda) \\
&\quad + a^*_4(-1+\lambda)(-24+\lambda(-7+v)(-6+v)) \\
c^*_{42} &= -6(c^*_{17} + 3c^*_{72}) + b^*_6(-18+\lambda(48+(-13+v)v)) \\
c^*_{43} &= 2(10b^*_{16} - 7b^*_6 + c^*_{20} - 7c^*_{72} + c^*_{89} - 2c^*_{90}) \\
&\quad + \lambda(b^*_{16}(-17+v) - b^*_6(-13+v) + (-1+\lambda)^2(6+\lambda(-9+v))(-1+v)v) \\
&\quad + a^*_4(34 + \lambda(-59 - 2\lambda(-12+v) + 3v)) \\
c^*_{44} &= 2(8b^*_{16} - 6b^*_6 - 2c^*_{17} + c^*_{21} - 6c^*_{72} + c^*_{89} - c^*_{90}) \\
&\quad - a^*_4(-3+2\lambda)(8+\lambda(-11+v)) + \lambda(b^*_{16}(-15+v) - b^*_6(-15+v) \\
&\quad + (-1+\lambda)^2(4+\lambda(-9+v))(-1+v)v)
\end{aligned}
$$

$$\begin{aligned}
c_{45}^* &= -2(4b_6^* + c_{17}^* - c_{20}^* - 2c_{21}^* + 4c_{72}^* + c_{90}^*) + b_{16}^*(4 + \lambda(-11 + v)) \\
&\quad + \lambda((-b_6^*)(-13 + v) + (-1 + \lambda)^2(2 + \lambda(-9 + v))(-1 + v)v) \\
&\quad + a_4^*(10 + \lambda(-37 - 2\lambda(-10 + v) + 3v)) \\
c_{46}^* &= 20b_{16}^* - 12b_6^* + c_{18}^* + 2c_{20}^* - 12c_{72}^* + c_{89}^* - 5c_{90}^* \\
&\quad + \lambda(-16b_{16}^* + 8b_6^* + (-1 + \lambda)^2(8 + \lambda(-10 + v))(-1 + v)v) \\
&\quad + a_4^*(42 + 2\lambda(-34 - \lambda(-14 + v) + v)) \\
c_{47}^* &= -4c_{17}^* + c_{19}^* + 2c_{21}^* - 12c_{72}^* + c_{89}^* - c_{90}^* + b_{16}^*(8 - 10\lambda) \\
&\quad + a_4^*(12 - \lambda(33 + \lambda(-20 + v))) \\
&\quad + (-1 + \lambda)(14b_6^* + \lambda(-2 + \lambda(10 + \lambda(-11 + v)))(-1 + v)v) \\
c_{48}^* &= 2b_{16}^* - 8b_6^* - c_{17}^* + c_{19}^* + c_{20}^* + 5c_{21}^* - 6c_{72}^* + c_{73}^* - c_{90}^* \\
&\quad + a_4^*(5 - \lambda(24 + \lambda(-18 + v))) \\
&\quad + \lambda(-7b_{16}^* + 11b_6^* + (-1 + \lambda)(-1 + \lambda(8 + \lambda(-11 + v)))(-1 + v)v) \\
c_{49}^* &= -2c_{17}^* + c_{18}^* + 3c_{19}^* + 4c_{21}^* - 4c_{72}^* - 2c_{90}^* - 8b_{16}^*(-1 + \lambda) \\
&\quad + (-1 + \lambda)(10b_6^* + (-1 + \lambda)\lambda(4 + \lambda(-10 + v))(-1 + v)v) \\
&\quad - a_4^*(-1 + \lambda)(20 + \lambda(-21 + 2v)) \\
c_{50}^* &= -2c_{17}^* + 4c_{21}^* - 10c_{72}^* + 2c_{73}^* - 2b_6^*(5 + \lambda(-10 + v)) \\
&\quad + \lambda(-4b_{16}^* - a_4^*(12 + \lambda(-13 + v)) \\
&\quad + (-1 + \lambda)\lambda(4 + \lambda(-9 + v))(-1 + v)v) \\
c_{51}^* &= -4c_{17}^* + 2c_{19}^* + 4c_{21}^* - 8c_{72}^* - 2b_6^*(6 + \lambda(-11 + v)) \\
&\quad + \lambda(-2b_{16}^* - a_4^*(8 + \lambda(-11 + v)) \\
&\quad + (-1 + \lambda)\lambda(4 + \lambda(-9 + v))(-1 + v)v) \\
c_{52}^* &= 2c_{19}^* + 8c_{21}^* + 2c_{73}^* - 2b_6^*(2 + \lambda(-9 + v)) \\
&\quad - \lambda(2 + \lambda(-9 + v))(a_4^* + \lambda v(-1 + \lambda + v - \lambda v)) \\
c_{53}^* &= c_{19}^* + 2c_{20}^* + 6c_{21}^* - c_{23}^* + b_6^*(-2 + 8\lambda) \\
&\quad + \lambda(-4b_{16}^* - a_4^*(13 + \lambda(-14 + v)) \\
&\quad + (-1 + \lambda)\lambda(5 + \lambda(-10 + v))(-1 + v)v) \\
c_{54}^* &= c_{18}^* + 4(c_{19}^* + c_{21}^*) - c_{23}^* - 4b_{16}^*(-1 + \lambda) \\
&\quad - 2a_4^*(-1 + \lambda)(5 + \lambda(-8 + v)) \\
&\quad + (-1 + \lambda)(8b_6^* + (-1 + \lambda)\lambda(2 + \lambda(-9 + v))(-1 + v)v) \\
c_{55}^* &= -6(16b_{16}^* + c_{18}^* + 2(c_{19}^* - 4c_{72}^* + c_{89}^* - 2c_{90}^*)) \\
&\quad + 2b_{16}^*\lambda(108 + (-19 + v)v) \\
&\quad + a_4^*(-120 + \lambda(486 - 82v + 4v^2 + \lambda(-411 + v(155 + (-21 + v)v))))
\end{aligned}$$

$$
\begin{aligned}
c_{56}^* &= 2(-6c_{18}^* - 12c_{19}^* + c_{22}^* + 12c_{72}^* + 6c_{90}^* - 6b_{16}^*(4 + \lambda(-11 + v))) \\
&\quad + a_4^*(-96 + 2\lambda(-15 + v)(-13 + v) + \lambda^2(-7 + v)(57 + (-14 + v)v)) \\
c_{57}^* &= -6(10b_{16}^* - 9b_6^* - 2c_{17}^* + c_{20}^* + 2c_{21}^* - 9c_{72}^* + c_{89}^* - 2c_{90}^*) \\
&\quad + a_4^*(-102 - 2\lambda^2(84 + (-16 + v)v) + 3\lambda(107 + (-16 + v)v)) \\
&\quad + \lambda(b_{16}^*(99 + (-16 + v)v) \\
&\quad - b_6^*(99 + (-16 + v)v) + (-1 + \lambda)^2(-1 + v)v(-18 + \lambda(75 + (-16 + v)v))) \\
c_{58}^* &= 64b_6^* - 2(c_{18}^* + 4c_{20}^* - 32c_{72}^* + 6c_{89}^* - 12c_{90}^*) - 8b_{16}^*(16 + \lambda(-15 + v)) \\
&\quad - a_4^*(224 + \lambda(-428 + 28v + \lambda(203 + (-28 + v)v))) + \lambda(4b_6^*(-15 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(40 + \lambda(2(-57 + 5v) + \lambda(89 + (-18 + v)v)))) \\
c_{59}^* &= -2(48b_{16}^* - 32b_6^* - 8c_{17}^* + c_{19}^* + 4c_{21}^* - 30c_{72}^* + 6c_{89}^* - 6c_{90}^*) \\
&\quad - a_4^*(144 + \lambda(-338 + 24v + \lambda(183 + (-26 + v)v))) \\
&\quad + \lambda(4b_6^*(-19 + v) - 8b_{16}^*(-13 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(24 + \lambda(-94 + 8v + \lambda(91 + (-18 + v)v)))) \\
c_{60}^* &= 52b_6^* + 8c_{17}^* - 2(c_{19}^* + 2(c_{20}^* + 4c_{21}^* - 12c_{72}^* + c_{73}^* + c_{89}^* - 2c_{90}^*)) \\
&\quad + b_{16}^*(-40 + \lambda(74 - 6v)) - a_4^*(68 + 6\lambda(-40 + 3v) \\
&\quad + \lambda^2(161 + (-24 + v)v)) + \lambda(b_6^*(-82 + 6v) \\
&\quad + (-1 + \lambda)(-1 + v)v(12 + \lambda(6(-12 + v) + \lambda(91 + (-18 + v)v)))) \\
c_{61}^* &= -2(20b_{16}^* - 26b_6^* - 4c_{17}^* + c_{18}^* + 3c_{19}^* + 2c_{20}^* + 8c_{21}^* - 20c_{72}^* + c_{73}^* + c_{89}^* \\
&\quad - 5c_{90}^*) - a_4^*(84 + \lambda(-262 + 16v + \lambda(171 + (-24 + v)v))) \\
&\quad + \lambda(2b_6^*(-33 + v) - 4b_{16}^*(-18 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(16 + \lambda(-82 + 6v + \lambda(95 + (-18 + v)v)))) \\
c_{62}^* &= 12c_{17}^* - 6c_{19}^* - 24c_{21}^* + 36c_{72}^* - 6c_{73}^* - 2b_6^*(-24 + \lambda(81 + (-16 + v)v)) \\
&\quad + \lambda(12b_{16}^* + (-1 + \lambda)\lambda(-1 + v)v(-18 + \lambda(75 + (-16 + v)v)) \\
&\quad + a_4^*(42 - \lambda(87 + (-16 + v)v))) \\
c_{63}^* &= -2(24b_{16}^* - 28b_6^* - 8c_{17}^* + c_{18}^* + 4c_{19}^* + 8c_{21}^* - 20c_{72}^* + 2c_{89}^* - 4c_{90}^*) \\
&\quad - a_4^*(88 + 16\lambda(-14 + v) + \lambda^2(151 + (-24 + v)v)) \\
&\quad + \lambda(-4b_{16}^*(-15 + v) + 8b_6^*(-12 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(16 + \lambda(-70 + 6v + \lambda(89 + (-18 + v)v)))) \\
c_{64}^* &= -8b_{16}^* + 32b_6^* + 8c_{17}^* - 4c_{19}^* - 8c_{20}^* - 24c_{21}^* + 2c_{23}^* + 24c_{72}^* + 4c_{90}^* \\
&\quad - a_4^*(20 + 12\lambda(-12 + v) + \lambda^2(133 + (-22 + v)v)) \\
&\quad + \lambda(-4b_{16}^*(-11 + v) + 8b_6^*(-11 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(4 + \lambda(4(-12 + v) + \lambda(89 + (-18 + v)v))))
\end{aligned}
$$

$$
\begin{aligned}
c^*_{65} &= -2(8b^*_{16} - 18b^*_6 - 2c^*_{17} + c^*_{18} + 5c^*_{19} + 2c^*_{20} + 12c^*_{21} - c^*_{23} - 8c^*_{72} + c^*_{73} \\
&\quad - 2c^*_{90}) - a^*_4(40 + 2\lambda(-82 + 5v) + \lambda^2(141 + (-22 + v)v)) \\
&\quad + \lambda(-2b^*_{16}(-21 + v) + 4b^*_6(-18 + v) \\
&\quad + (-1 + \lambda)(-1 + v)v(8 + \lambda(-58 + 4v + \lambda(93 + (-18 + v)v)))) \\
c^*_{66} &= 4c^*_{17} - 7c^*_{18} - 20c^*_{19} - 16c^*_{21} + c^*_{22} + 2c^*_{23} + 8c^*_{72} + 4c^*_{90} \\
&\quad + 48b^*_{16}(-1 + \lambda) - 2a^*_4(-1 + \lambda)(-60 + \lambda(88 + (-17 + v)v)) \\
&\quad + (-1 + \lambda)(-48b^*_6 + (-1 + \lambda)\lambda(-1 + v)v(-24 + \lambda(86 + (-17 + v)v))) \\
c^*_{67} &= a^*_4(648 - 6\lambda(522 + 5(-19 + v)v) \\
&\quad + \lambda^2(-9 + v)(-396 + v(145 + (-20 + v)v))) \\
&\quad - 6(-9c^*_{18} - 18c^*_{19} + c^*_{22} + 36c^*_{72} - 6c^*_{89} + 18c^*_{90} \\
&\quad + 2b^*_{16}(-36 + \lambda(108 + (-19 + v)v))) \\
c^*_{68} &= 6(64b^*_{16} - 56b^*_6 - 8c^*_{17} + c^*_{18} + 3c^*_{19} \\
&\quad + 2(2c^*_{20} + 6c^*_{21} - 25c^*_{72} + c^*_{73} + 3c^*_{89} - 6c^*_{90})) \\
&\quad + a^*_4(672 - \lambda(6(379 + 2(-23 + v)v) + \lambda(-1512 + v(335 + (-30 + v)v)))) \\
&\quad + \lambda(2b^*_6(258 + (-25 + v)v) - 4b^*_{16}(171 + (-22 + v)v) \\
&\quad + (-1 + \lambda)(-1 + v)v(-120 + \lambda(684 - 94v + 4v^2 \\
&\quad + \lambda(-846 + v(241 + (-26 + v)v))))) \\
c^*_{69} &= 6(28b^*_{16} - 44b^*_6 - 8c^*_{17} + c^*_{18} + 6c^*_{19} + 4c^*_{20} + 20c^*_{21} - c^*_{23} \\
&\quad + 2(-16c^*_{72} + c^*_{73} + c^*_{89} - 3c^*_{90})) \\
&\quad + a^*_4(324 - \lambda(6(238 + (-27 + v)v) + \lambda(-1266 + v(297 + (-28 + v)v)))) \\
&\quad + \lambda(-2b^*_{16}(210 + (-25 + v)v) + 4b^*_6(165 + (-22 + v)v) \\
&\quad + (-1 + \lambda)(-1 + v)v(-60 + 2\lambda(231 + (-28 + v)v) \\
&\quad + \lambda^2(-840 + v(241 + (-26 + v)v)))) \\
c^*_{70} &= -2(136b^*_6 + 16c^*_{17} - 11c^*_{18} - 36c^*_{19} - 8c^*_{20} - 56c^*_{21} + c^*_{22} + 4c^*_{23} + 64c^*_{72} \\
&\quad - 4c^*_{73} - 2c^*_{89} + 18c^*_{90}) + 8b^*_{16}(26 + 3\lambda(-17 + v)) \\
&\quad + a^*_4(488 - \lambda(4(424 + (-39 + v)v) + \lambda(-1396 + v(311 + (-28 + v)v)))) \\
&\quad + \lambda(-24b^*_6(-19 + v) + (-1 + \lambda)(-1 + v)v(-96 + 2\lambda(292 + (-33 + v)v) \\
&\quad + \lambda^2(-884 + v(245 + (-26 + v)v)))) \\
c^*_{71} &= 6(336b^*_6 + 48c^*_{17} - 15c^*_{18} - 54c^*_{19} - 24c^*_{20} - 120c^*_{21} + c^*_{22} \\
&\quad + 6(c^*_{23} + 38c^*_{72} - 2c^*_{73} - 3c^*_{89} + 9c^*_{90})) + 24b^*_{16}(-72 + \lambda(171 + (-22 + v)v)) \\
&\quad - a^*_4(3456 - 12\lambda(1209 + v(-157 + 7v)) \\
&\quad + \lambda^2(13104 + v(-3825 + v(511 + (-35 + v)v)))) \\
&\quad + \lambda(-24b^*_6(183 + (-22 + v)v) \\
&\quad + (-1 + \lambda)(-1 + v)v(648 - 6\lambda(786 + v(-113 + 5v)) \\
&\quad + \lambda^2(8712 + v(-3147 + v(481 + (-35 + v)v)))))
\end{aligned}
$$

$$
\begin{aligned}
c^*_{74} &= 2c^*_{89} + b^*_{16}(6 + \lambda(-5 + v)) \\
c^*_{75} &= -c^*_{72} + c^*_{89} + b^*_{16}(5 - 2\lambda) + a^*_4(-2 + \lambda)^2 \\
c^*_{76} &= b^*_{16} - c^*_{72} - c^*_{90} + a^*_4(-2 + \lambda)(-1 + \lambda) - b^*_{16}\lambda \\
c^*_{77} &= -2c^*_{72} + (-b^*_{16} + a^*_4(-2 + \lambda))\lambda \\
c^*_{78} &= -2(b^*_6 + c^*_{72}) + b^*_6\lambda \\
c^*_{79} &= -6c^*_{89} + b^*_{16}(-18 + \lambda(27 + (-10 + v)v)) \\
c^*_{80} &= 2(c^*_{72} - c^*_{89} + c^*_{90}) - b^*_{16}(12 + \lambda(-9 + v)) \\
&\quad + a^*_4(-2 + \lambda)(6 + \lambda(-7 + v)) \\
c^*_{81} &= 4c^*_{72} - 2c^*_{89} - b^*_{16}(10 + \lambda(-9 + v)) \\
&\quad + a^*_4(-2 + \lambda)(4 + \lambda(-7 + v)) \\
c^*_{82} &= 2(2c^*_{72} + c^*_{90}) - b^*_{16}(2 + \lambda(-7 + v)) \\
&\quad + a^*_4(-2 + \lambda)(2 + \lambda(-7 + v)) \\
c^*_{83} &= 4c^*_{72} - c^*_{89} + c^*_{90} + 4b^*_{16}(-2 + \lambda) \\
&\quad - 2a^*_4(-2 + \lambda)(-3 + 2\lambda) + (-2 + \lambda)(-2b^*_6 + (-1 + \lambda)^2\lambda(-1 + v)v) \\
c^*_{84} &= 4c^*_{72} - c^*_{73} - 2b^*_6(-2 + \lambda) \\
&\quad + \lambda(2b^*_{16} + (-2 + \lambda)(-3a^*_4 + (-1 + \lambda)\lambda(-1 + v)v)) \\
c^*_{85} &= 6(6b^*_{16} - 2c^*_{72} + c^*_{89} - c^*_{90}) - b^*_{16}\lambda(54 + (-13 + v)v) \\
&\quad + a^*_4(-2 + \lambda)(-18 + \lambda(48 + (-13 + v)v)) \\
c^*_{86} &= -6(2c^*_{72} - c^*_{89} + c^*_{90}) + 2b^*_{16}(24 + \lambda(-15 + v)) \\
&\quad - 3a^*_4(-2 + \lambda)(12 + \lambda(-11 + v)) \\
&\quad + (-2 + \lambda)(6b^*_6 + (-1 + \lambda)\lambda(6 + \lambda(-9 + v))(-1 + v)v) \\
c^*_{87} &= 2(-8c^*_{72} + c^*_{73} + c^*_{89} - c^*_{90}) + 2b^*_{16}(8 + \lambda(-11 + v)) \\
&\quad + (-2 + \lambda)(8b^*_6 + (-1 + \lambda)\lambda(2 + \lambda(-9 + v))(-1 + v)v) \\
&\quad - a^*_4(-2 + \lambda)(12 + \lambda(-29 + 3v)) \\
c^*_{88} &= 6(12c^*_{72} - c^*_{73} - 3c^*_{89} + 3c^*_{90}) - 3a^*_4(-2 + \lambda)(-36 + \lambda(81 + (-16 + v)v)) \\
&\quad + 2b^*_{16}(-72 + \lambda(93 + (-16 + v)v)) \\
&\quad + (-2 + \lambda)(-36b^*_6 + (-1 + \lambda)\lambda(-1 + v)v(-18 + \lambda(75 + (-16 + v)v))) \\
c^*_{91} &= -c^*_{89} + b^*_{16}(-3 + \lambda) \\
c^*_{92} &= c^*_{89} - c^*_{90} - 2b^*_{16}(-3 + \lambda) \\
&\quad + a^*_4(-3 + \lambda)(-2 + \lambda) \\
c^*_{93} &= -3c^*_{89} + 3c^*_{90} - 6a^*_4(-3 + \lambda)(-2 + \lambda) \\
&\quad + (-3 + \lambda)(8b^*_{16} + (-2 + \lambda)(-1 + \lambda)\lambda(-1 + v)v)
\end{aligned}
$$

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